Classical simulation of quantum circuits with partial and graphical stabiliser decompositions

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TQC – July 13th 2022

Quantum circuit simulation

The problem: (somewhat) **efficiently** simulating quantum circuits.

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  - direct state simulation
  - principle component analysis
  - contraction order finding
  - edge cutting
  - ...

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Broadly, two ways to do this:

- Tensor-network based methods:
  - direct state simulation
  - principle component analysis
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  - ...
- Stabiliser-decomposition based methods:
  - Stabiliser extent
  - Stabiliser rank $\leftarrow$ **This talk**

Simulating using stabiliser rank

- Start with Clifford+T circuit.
- Write each T gate as magic state injection.
- Decompose T states into sum of stabilisers.
- Efficiently simulate resulting Clifford circuits.
- Add results together.
- We're done!

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What's the catch?
Stabiliser rank of k T states scales exponentially with k.
... But it's not just 2^k terms. We can do better!

Stabiliser ranks of T magic states

Recall $|T\rangle \;\propto\; |0\rangle + e^{i\pi/4}|1\rangle$.
So $\chi(|T\rangle) = 2$ and hence $\chi(|T\rangle^{\otimes k}) \leq 2^k$.

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Turns out $\chi(|T\rangle^{\otimes 6}) \leq 7$ so
$\chi(|T\rangle^{\otimes k}) \leq 2^{\alpha k}$ where $\alpha = \log_2(7)/6 \approx 0.467$.
Found by Bravyi, Smith, Smolin (BSS) in 2016.

Even have $\chi(|T\rangle^{\otimes 6}) = 6$ (previous talk): $\alpha \approx 0.431$.

Our idea

Do stabiliser rank decompositions,
but with *ZX-diagrams* instead of circuits!

Benefit 1: optimise intermediate ZX-diagrams to reduce T-count.
Benefit 2: Can use *fancier* stabiliser decompositions.

Spiders

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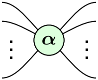
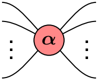
Z-spider

$$|0\cdots0\rangle\langle0\cdots0| + e^{i\alpha}|1\cdots1\rangle\langle1\cdots1|$$



X-spider

$$|+\cdots+\rangle\langle+\cdots+| + e^{i\alpha}|-\cdots-\rangle\langle-\cdots-|$$



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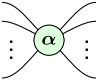

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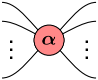
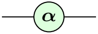
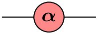


For example:



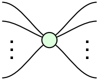
$$= |0\rangle\langle 0| + e^{i\alpha}|1\rangle\langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & e^{i\alpha} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$



$$= |+\rangle\langle +| + e^{i\alpha}|-\rangle\langle -| = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{1}{2}e^{i\alpha}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Spiders cont.

If $\alpha = 0$ we drop the label:



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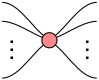

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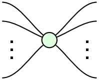

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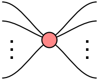

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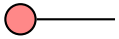
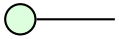
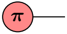
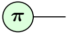
Example:

 $= \quad |+\rangle + |-\rangle \ = \sqrt{2}|0\rangle$

 $= \quad |0\rangle + |1\rangle \ = \sqrt{2}|+\rangle$



$= \quad |+\rangle - |-\rangle \ = \sqrt{2}|1\rangle$



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Formal composition

Spiders can be composed in two ways.

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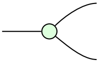
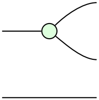
Vertical composition gives tensor product:



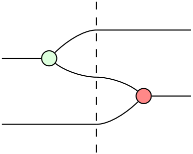
$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$



$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Formal composition

Horizontal composition is regular composition of linear maps:

 $=$

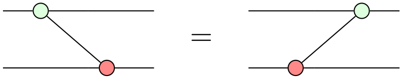
$$\frac{1}{\sqrt{2}}\begin{pmatrix}1&0&0&1&0&0&0&0\\0&1&1&0&0&0&0&0\\0&0&0&0&1&0&0&1\\0&0&0&0&0&1&1&0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$$

Building ZX-diagrams

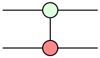
Any ZX-diagram is built by simply iterating these vertical and horizontal compositions

Symmetries

Note:



Hence, we may write



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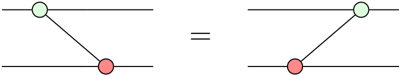

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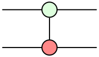
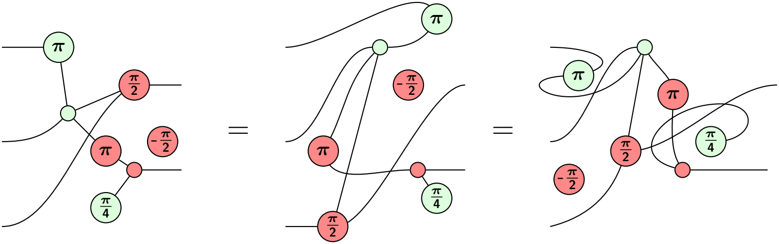


In general: *only connectivity matters*



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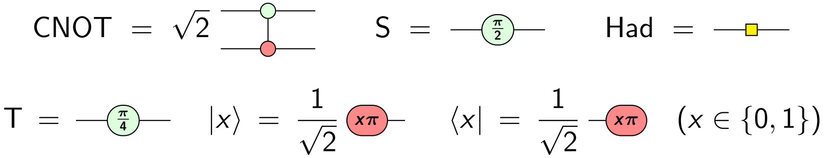
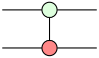
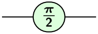
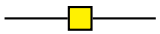
Let's go through these steps in more detail.

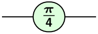
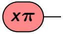
- Writing circuit as ZX-diagram
- Optimising ZX-diagram
- Decomposing magic states

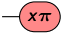
Writing Clifford+T circuit as ZX-diagram



Writing Clifford+T circuit as ZX-diagram

CNOT $= \sqrt{2}$  S $=$  Had $=$ 

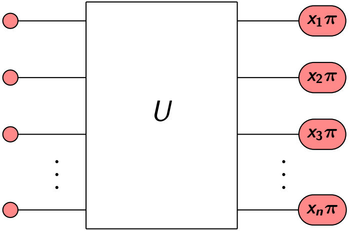
T $=$  $|x\rangle = \frac{1}{\sqrt{2}}$ 

$\langle x| = \frac{1}{\sqrt{2}}$ 

$(x \in \{0,1\})$

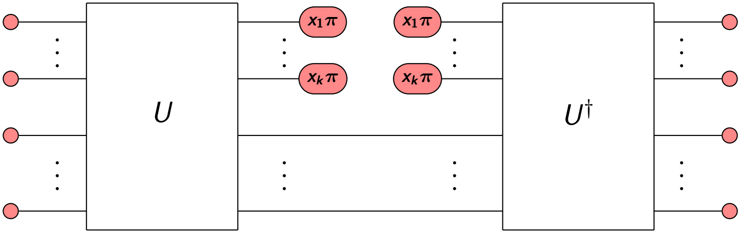
Calculating single amplitude:

$$\langle \vec{x}|U|0\cdots 0\rangle = \left(\frac{1}{\sqrt{2}}\right)^{2n}$$



Marginal probabilities

To calculate marginal probability, use *doubling* technique:



Strong simulation vs Weak simulation

Weak sim: approx. sample from the same output distribution.
Strong sim: approx. calculate any marginal probability.

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We are doing *exact* strong simulation here.

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Simplifying ZX-diagrams

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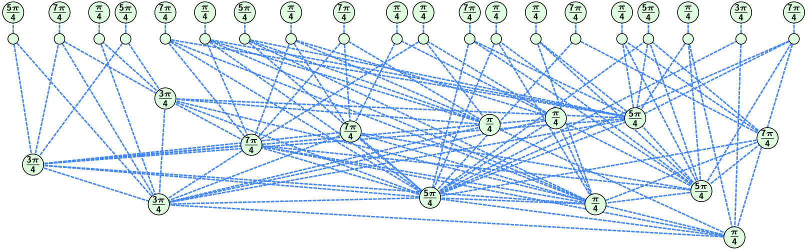
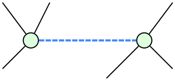
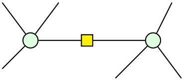
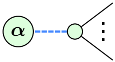
Reducing T-count with the ZX-calculus.
This transforms every ZX-diagram into something like this:



Here  := 

Properties of reduced diagram

The important part:

- Every spider carries a non-Clifford phase,
- or is part of a *phase gadget*: 

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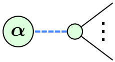
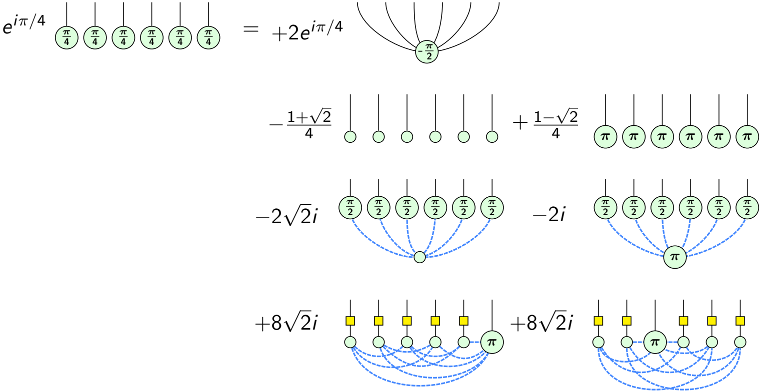
- or is part of a *phase gadget*: 

Particularly: if original circuit had k T gates, resulting diagram has $\leq 2k$ spiders (regardless of #qubits or #gates).

- Writing circuit as ZX-diagram ✓
- Optimising ZX-diagram ✓
- Decomposing magic states

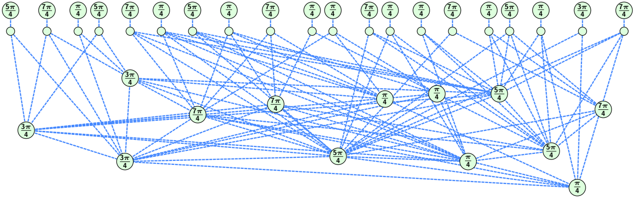
Decomposing T-like spiders

The 6-to-7 magic state decomposition in ZX is:



Applying the decomposition

We pick some spiders to decompose and unfuse the phases:



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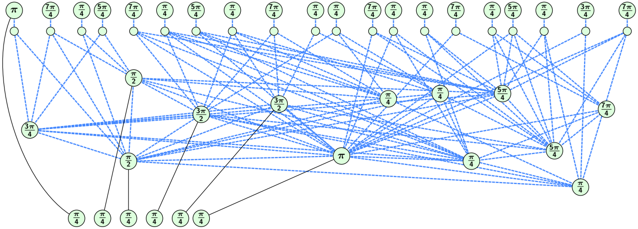

And now we can apply the magic state decomposition.

Better decompositions

Improved upper bounds on the stabilizer rank of magic states

Hammam Qassim [*†‡§] Hakop Pashayan [*¶‖] David Gosset [*¶]

June 16, 2021

$\Rightarrow$ improved stabiliser decompositions, including 6-to-6 decomp (giving $\alpha \approx 0.431$ instead of $\alpha \approx 0.467$), and other decomps giving $\alpha < 0.40$.

Cat states

Qassim et al. uses *cat states*:

$$|\,\mathrm{cat}_n\rangle := \frac{1}{\sqrt{2}}(\mathbb{I}^{\otimes n} + Z^{\otimes n})|\,T\rangle^{\otimes n} = \frac{1}{\sqrt{2}^{n+1}}\left(\textcircled{\tiny $\frac{\pi}{4}$}\!-\!\!-^{\otimes n} + \textcircled{\tiny $\frac{5\pi}{4}$}\!-^{\otimes n} \right)$$

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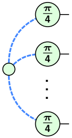
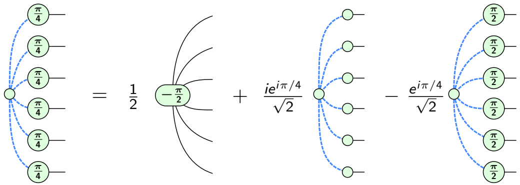
These actually have nice representation in ZX:

$$|\,\mathrm{cat}_n\rangle = \frac{1}{\sqrt{2}}$$



Cat decompositions

They find good decomp of $|\mathrm{cat}_6\rangle$:



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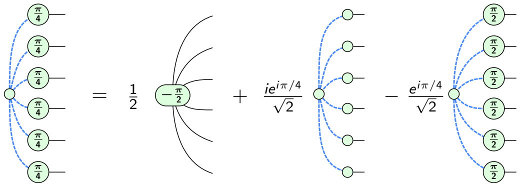

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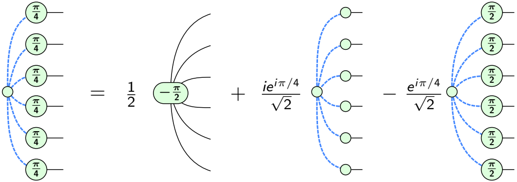
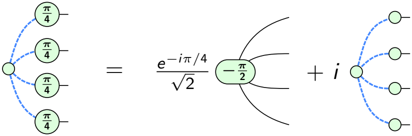


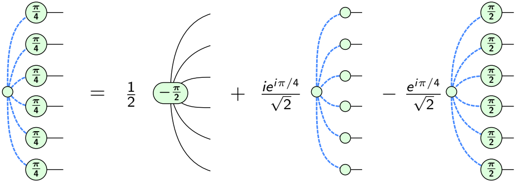
So, if we only have $|\text{cat}_6\rangle$ diagrams, we would have $\alpha \approx 0.264$.
But we can do even better!



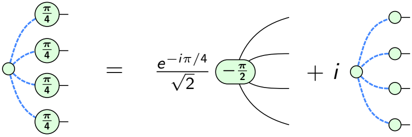
This would give $\alpha = 0.25$.

Cat decompositions

They find good decomp of $|\text{cat}_6\rangle$:



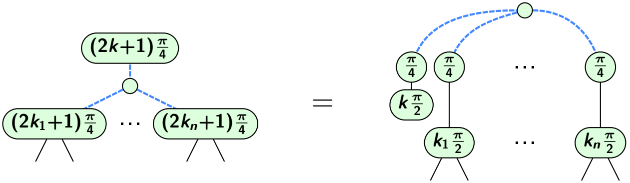
So, if we only have $|\text{cat}_6\rangle$ diagrams, we would have $\alpha \approx 0.264$.
But we can do even better!



This would give $\alpha = 0.25$.
Using these we get good decompositions for $|\text{cat}_k\rangle$ with $k \leq 6$.

Phase gadgets are cat states

We find cat states as phase gadgets:



So n-legged phase gadget is $|\mathrm{cat}_{n+1}\rangle$ state.

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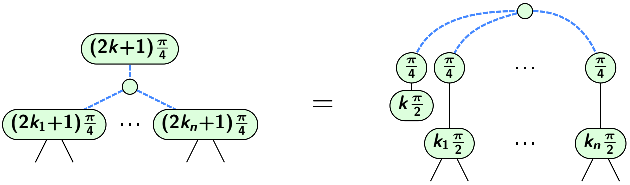


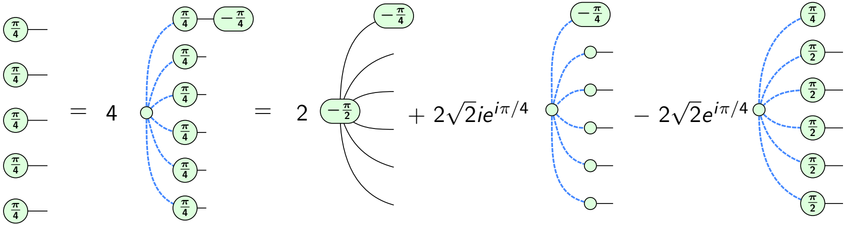
So n-legged phase gadget is $|\mathrm{cat}_{n+1}\rangle$ state.

So as long as there are phase gadgets with ≤ 5 legs, we can use these decompositions.

Partial stabiliser decomp

But what if there are no phase gadgets?

Then we can do the following ‘partial’ decomp:



This trades 5 magic states for 3 terms with 1 magic per term.
So effectively removes 4 magic states.
This is then a 4-to-3 decomp: $\alpha \approx 0.396$.

Full strategy

We are hence looking for the following things to decompose:

1. a phase gadget with 3 legs ($\alpha = 0.25$),
2. a phase gadget with 5 legs ($\alpha \approx 0.264$),
3. a phase gadget with 4 legs ($\alpha \approx 0.317$),
4. a phase gadget with 2 legs ($\alpha = 1/3 \approx 0.333$),
5. any 5 T-spiders ($\alpha \approx 0.396$).

So how well does all this work?

Asymptotic benefit

- Worst case: no T-like phases killed during simplification.

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- Size of diagrams are $O(k)$, so a rewrite costs $O(k^2)$.
- There are $O(2^{\alpha k})$ diagrams, so total cost is $O(2^{\alpha k} k^2)$.
- (Bravyi et al., 2016) gave $O(2^{\alpha k} k^3)$.
- Benefit comes from preventing 'double work': we 'partially evaluate' the stabilisers by simplifying the diagrams.

Actual benefit

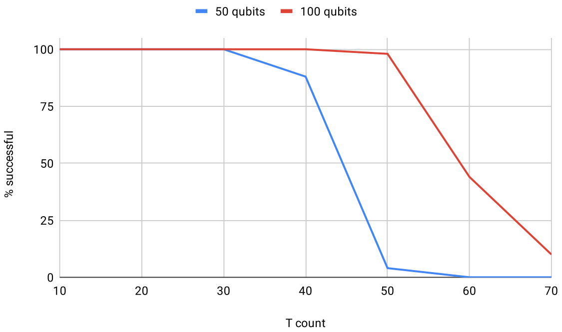
We benchmarked our method on two families of circuits:

- 50- and 100-qubit random Clifford+T circuits built out of Pauli exponentials.
- 50-qubit hidden-shift circuits (type of CCZ circuit).

We are sampling from the output distribution
(using strong simulation).

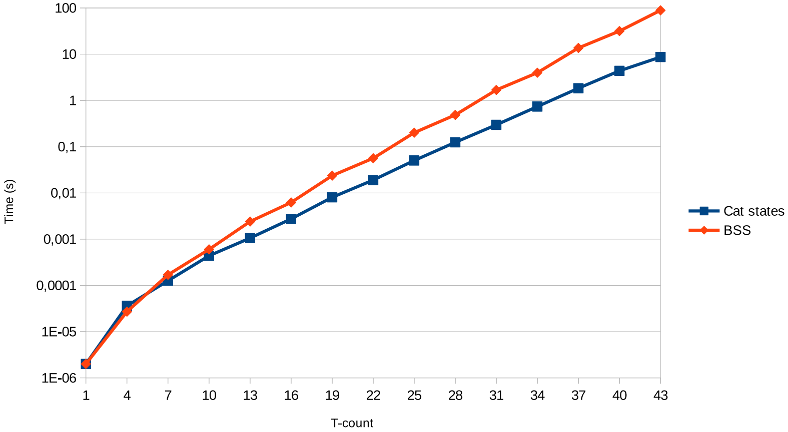
Code is implemented in `quizx`, a Rust port of `PyZX`.

Benchmark: Clifford+T



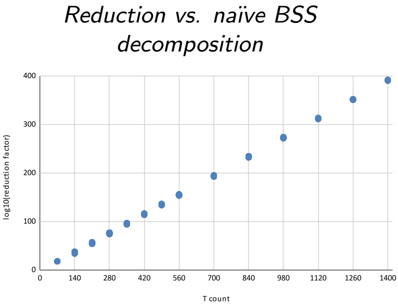
Percentage of random 50- and 100-qubit circuits of a given T-count that were successfully sampled in under 5 minutes. For each T-count 50 random circuits were generated.

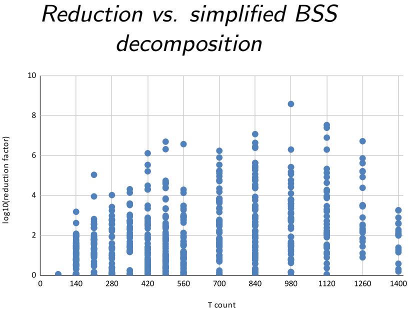
Benchmark: Clifford+T cat-decomp comparison



Runtime of random 20-qubit Clifford+T circuit simulations (avg of 10 runs per T-count).

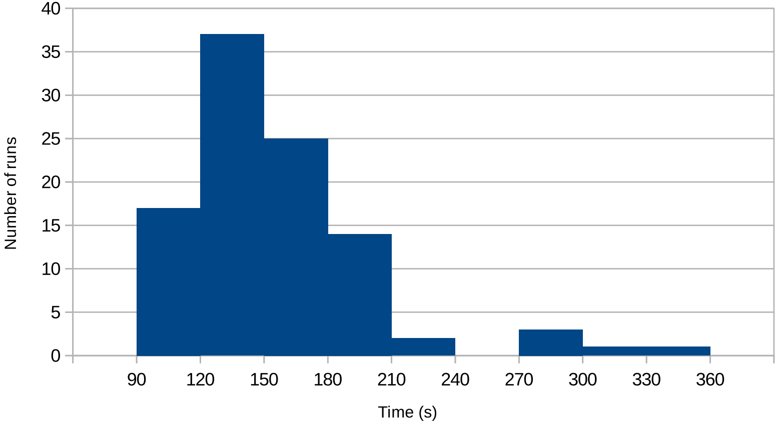
Benchmark: hidden-shift circuit term reduction





Reduction in term count on 50-qubit hidden shift circuits vs. naïve BSS decomposition (left) and BSS decomposition after single ZX-simplification (right).

Benchmark: hidden-shift 50-qubit simulation time



The time distribution of simulating 100 random 50-qubit hidden-shift circuits with T-count 1400 using our new decompositions.

Conclusions

- Using ZX we can greatly speed-up stabiliser rank simulations.
- Especially for structured circuits.
- It allows us to use better decompositions for substructures of diagrams, and to introduce partial stabiliser decompositions.

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Moral of the story: optimisation and simulation are not separate. They are two sides of the same coin.

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- Especially for structured circuits.
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Moral of the story: optimisation and simulation are not separate. They are two sides of the same coin.

Future work:

- Use more diagram optimisations and decompositions
- Find heuristics for picking good spiders to decompose.
- Approximate simulation and better weak simulation.
- Use *quantum measurement w/o computing marginals* technique.

Thank you for your attention!

Further reading:

- Kissinger & vdW. *Simulating quantum circuits with ZX-calculus reduced stabiliser decompositions*. arXiv: 2109.01076
- Kissinger, Vilmart & vdW. *Classical simulation of quantum circuits with partial and graphical stabiliser decompositions*. arXiv: 2202.09202
- Qassim, Pashayan, Gosset. *Improved upper bounds on the stabilizer rank of magic states*. arXiv: 2106.07740
- Bravyi, Gosset. *Improved classical simulation of quantum circuits dominated by Clifford gates*. arXiv: 1601.07601