Categorical approaches to reconstructing quantum theory

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Motivation

- Quantum theory is the mathematical framework for understanding microscopic reality.
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Can we derive quantum theory using category theory?

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- Schrödinger equation: $|\psi(t)\rangle = e^{-itH}|\psi\rangle$.

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- Why is a composite system described by a tensor product?

Some have good answers

Given we know that states are unit vectors in a Hilbert space:

- Gleason's theorem tells us why measurement updating works the way it does.
- Stone's theorem on one-parameter unitary groups gives Schrödinger equation.
- Principle of *local tomography* gives tensor product.

Core question: why complex Hilbert spaces?

In this talk

- Categorical characterisation of **Hilb** by Heunen and Kornell.
- Characterisation of completely-positive maps by Selinger & Coecke.
- Characterisation of CPM(**fHilb**) by Tull.
- Characterisation of probabilities $[0, 1]$ by Westerbaan, Westerbaan, vdW.

Hilbert spaces

Definition

A complex vector space $\mathcal{H}$ is a **Hilbert space** when it has

- an inner-product $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$,
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Note: as category $\mathbf{fHilb} \cong \mathbf{fVect}_{\mathbb{C}}$.
This is because we aren't capturing the inner product.

Dagger-categories

Definition

Cat $\mathbf{C}$ is $\dagger$-category when it has functor $\dagger : \mathbf{C} \to \mathbf{C}^{\mathrm{op}}$ satisfying $\dagger(A) = A$ and $\dagger^2 = \mathrm{id}$.

Concretely dagger of $f : A \to B$ is a $f^\dagger : B \to A$ such that $(f^\dagger)^\dagger = f$ and $(f \circ g)^\dagger = g^\dagger \circ f^\dagger$.

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Bounded maps $A : \mathcal{H} \to \mathcal{K}$ have unique *adjoint* $A^\dagger : \mathcal{K} \to \mathcal{H}$ satisfying $\langle Av, w \rangle_{\mathcal{K}} = \langle v, A^\dagger w \rangle_{\mathcal{H}}$. Makes **Hilb** into $\dagger$-category.

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For $v \in \mathcal{H}$ we have $\overline{v} : \mathbb{C} \to \mathcal{H}$, and then $\langle v, w \rangle_{\mathcal{H}} = \overline{v}^\dagger(w)$.

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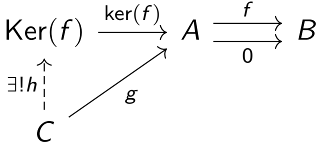

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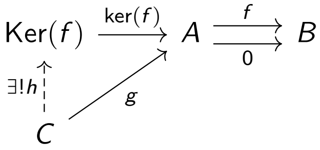

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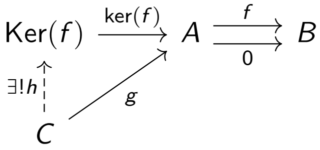

Can we characterise **Hilb** uniquely by these properties?
No: because **Hilb**×**Hilb** has the same structure.

Simple monoidal unit

In **Hilb** the monoidal unit $I := \mathbb{C}$ is

- simple: it has exactly two subobjects, i.e.
  if $f : \mathcal{H} \to I$ is mono, then $\mathcal{H} \cong 0$ or $\mathcal{H} \cong I$.

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Lemma (Heunen, 2009)

The scalars $\mathbf{C}(I, I)$ in a †-category $\mathbf{C}$ satisfying all the previous properties form an involutive field.

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So is this enough?
No, because $\mathbf{fHilb}_{\mathbb{Q}}$ also satisfies these properties.

A final axiom

The wide subcategory of †-mono's of **Hilb** has *directed colimits*.
An increasing net of Hilb spaces $\{\mathcal{H}_i\}_{i \in I}$, $\mathcal{H}_i \hookrightarrow \mathcal{H}_j$ for $i \leqslant j$, has a 'union' Hilbert space $\mathcal{H}_i \hookrightarrow \mathcal{H}$.

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Note: not true in **fHilb** as $\mathbb{C} \hookrightarrow \mathbb{C}^2 \hookrightarrow \mathbb{C}^3 \hookrightarrow \cdots$ has colimit $L^2(\mathbb{N})$.

Characterisation of **Hilb**

Definition (Heunen & Kornell 2021)

A *Hilbert category* is a †-symmetric monoidal category with

- †-biproducts,
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Theorem (Heunen & Kornell 2021)

Any Hilbert category is equivalent to either **Hilb** or $\mathbf{Hilb}_{\mathbb{R}}$.

Proof sketch

- Scalars are field, so monoidal separation makes it a category of vector spaces.

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- Directed colimits give infinite-dimensional spaces.
- **Solèr's Theorem**: If infinite-dim space is orthomodular, then field is $\mathbb{H}, \mathbb{C}$ or $\mathbb{R}$.
- $\mathbb{H}$ is not commutative, so scalars must be $\mathbb{C}$ or $\mathbb{R}$.

Open questions

- How to characterise just $\mathbf{Hilb}_{\mathbb{C}}$ and not also $\mathbf{Hilb}_{\mathbb{R}}$?
- How to characterise **fHilb**?
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- Can you get rid of the requirement the unit be simple to get a category of (pre)sheaves over **Hilb**?
- How to characterise mixed quantum theory, which includes measurement and noise?

Mixed states

We will focus on **fHilb** now.

- Write $B(\mathcal{H}) := \{ A : \mathcal{H} \to \mathcal{H} \text{ bounded} \}$.
- Call $A \in B(\mathcal{H})$ *positive* when $\langle Av, v \rangle \geqslant 0$ for all v.
- A *density operator* is a positive $\rho \in B(\mathcal{H})$ with $\mathrm{tr}(\rho) = 1$.

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  $|\psi\rangle\langle\psi| \, (|\phi\rangle) = \langle \psi, \phi \rangle \, |\psi\rangle$.
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To work with mixed states in quantum theory, we use $B(\mathcal{H})$ instead of $\mathcal{H}$ directly.

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Definition
Call $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$ *completely positive* when
$\Phi \otimes \mathrm{id}_n : B(\mathcal{H} \otimes \mathbb{C}^n) \to B(\mathcal{K} \otimes \mathbb{C}^n)$ is positive for all n.

Write **CPM** for cat of fin.dim. Hilb spaces and completely pos maps.

Embedding the pure into the mixed

- Pure QT has maps $A : \mathcal{H} \to \mathcal{K}$.
- In mixed setting: $\hat{A} : B(\mathcal{H}) \to B(\mathcal{K})$ by $\hat{A}(C) = ACA^\dagger$.
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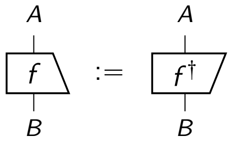
"Church of the Higher Hilbert Space"

Dagger-compact categories

Graphical notation for †-categories:



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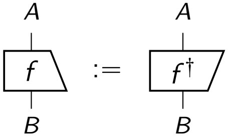
Graphical notation for †-categories: 

Definition

A †-category is †-*compact* when for all A there exists A^* and a state : $I \to A^* \otimes A$ satisfying the *snake equations*:

where wire with arrow pointing up is id_A, and the other is id_{A*}.

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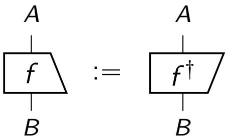

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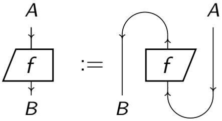

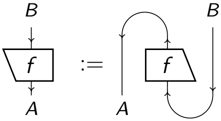

Matrix theories

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An *involutive semi-ring* S is a 'ring without negation', with an anti-automorphism satisfying $(s^\dagger)^\dagger = s$.
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$\mathbf{Mat}_S$ is $\dagger$-compact with $(M^\dagger)_{ij} = M_{ji}^\dagger$.
$\mathbf{Mat}_\mathbb{C} \cong \mathbf{fHilb}$.

Selinger's CPM construction

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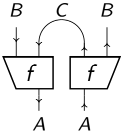
For †-compact **C** define CPM(**C**) as cat with same objects, but with morphisms



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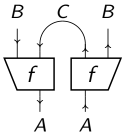

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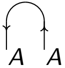
- The *doubling* functor $D : \mathbf{C} \to \mathrm{CPM}(\mathbf{C})$ maps f to $f \otimes f^*$.
- CPM(**C**) has *discard* map $\top_A$:= 

- Morphisms of CPM(**C**) are generated by $D(\mathbf{C})$ and discarding.

Hilbert space CPM construction

In the case of Hilbert spaces:

- We have CPM(**fHilb**) $\cong$ **CPM**,
- doubling functor gives the pure maps,
- discarding is the trace,
- doubling and trace generating the morphisms is Stinespring dilation.

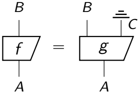
Hilbert space CPM construction

In the case of Hilbert spaces:

- We have CPM(**fHilb**) $\cong$ **CPM**,
- doubling functor gives the pure maps,
- discarding is the trace,
- doubling and trace generating the morphisms is Stinespring dilation.

Q: Can we identify when a category is of the form CPM(**C**)?

Environment structure

In †-cat with discarding, g is **dilation** of f when 

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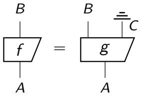
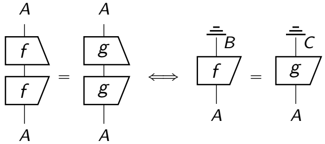
In †-cat with discarding, g is **dilation** of f when 

Definition (Coecke, 2008)

Let $\mathbf{C}$ be †-compact with discarding.
An *environment structure* is choice of subcat $\mathbf{C}_p$, such that

- Every f in $\mathbf{C}$ has dilation in $\mathbf{C}_p$.
- All f, g in $\mathbf{C}_p$ satisfy *CP-axiom*: 

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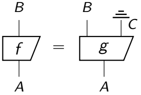
In †-cat with discarding, g is **dilation** of f when 

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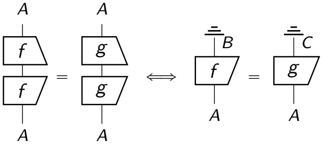
- All f, g in $\mathbf{C}_p$ satisfy *CP-axiom*: 

Theorem: $\mathrm{CPM}(\mathbf{C}_p) \cong \mathbf{C}$.

Tull's reconstruction of quantum theory

We will now give the axioms from Tull's paper *A categorical reconstruction of quantum theory*.

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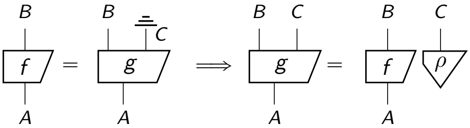
Informational derivation of quantum theory

Intuitively:

- Every map has an essentially unique purification.
- Kernels exist and are well-behaved.
- Every pure state can be perfectly distinguished from at least one other pure state.
- We can conditionally prepare states.

Pure maps and purification

Call a map f *pure* when $f = 0$, or any dilation of f is trivial:



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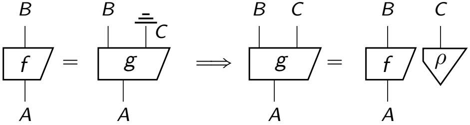
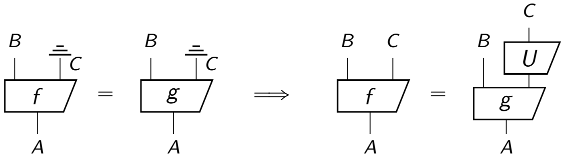


Axiom 1: Pure maps form environment structure, and pure dilations are *essentially unique*: for any f, g pure



for some †-iso U.

Kernels and complements

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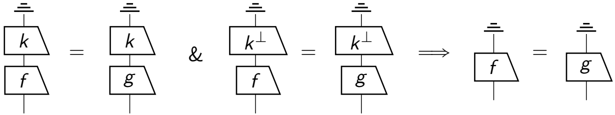
Axiom 2: The category has †-kernels which are *causally complemented*:



Pure exclusion

Call $f\colon A \to B$ *causal* when $\ground_B \circ f = \ground_A$.

Pure exclusion

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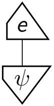
Axiom 3: Every non-zero object A has a causal pure state.
If furthermore $A \not\cong I$, then for all causal pure $\psi : I \to A$ there is a non-zero e such that



$= 0$

Conditioning

Call states $|0\rangle, |1\rangle : I \to A$ *orthonormal* when

$$|0\rangle^\dagger \circ |0\rangle = \mathrm{id}_I = |1\rangle^\dagger \circ |1\rangle \text{ and } |1\rangle^\dagger \circ |0\rangle = 0$$

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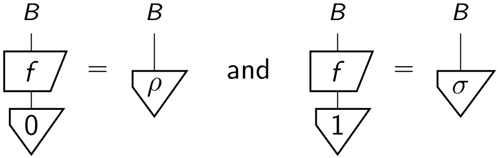
Axiom 4: For every orthonormal $|0\rangle, |1\rangle : I \to A$ and any $\rho, \sigma : I \to B$ there is $f : A \to B$ with



Matrix theories revisited

When does $\mathrm{CPM}(\mathbf{Mat}_S)$ satisfy these assumptions for an involutive semi-ring S?

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Definition

A **phased ring** is a ring

- which is commutative,
- with involution $(a^\dagger)^\dagger = a$,
- having no zero-divisors: $a \cdot b = 0 \implies a = 0$ or $b = 0$,
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Prop: $\mathrm{CPM}(\mathbf{Mat}_S)$ satisfies the assumptions iff S is phased ring.

Open question: What are the phased rings? Are they always a field?

We say **C** is a *Tull-category* when it is †-compact, non-trivial,

- has zero morphisms,
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- can perfectly distinguish non-trivial pure states,
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Theorem (Tull,2019)

Let $\mathbf{C}$ be a Tull-category with scalars $R = \mathbf{C}(I, I)$.
Then there is an embedding $\mathrm{CPM}(\mathbf{Mat}_S) \hookrightarrow \mathbf{C}$
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Corollary: If $\mathbf{C}$ is a Tull-category with $\mathbf{C}(I, I) = \mathbb{R}_{\geqslant 0}$,
then $\mathbf{C} \cong \mathbf{CPM}$ or $\mathbf{C} \cong \mathbf{CPM}_{\mathbb{R}}$.

Deriving real numbers

- So why real numbers?

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- So why are probabilities real numbers?
- Can we derive the structure of $[0, 1]$ from abstract grounds?

Structure of probabilities

- (Partially-defined) addition for coarse-graining.
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- Some kind of limiting behaviour. Gives you *ω-effect monoids*.

This turns out to be enough to construct real numbers.

Addition and coarse-graining

Suppose we have a probability distribution $P(X = x_i)$ for $x_i \in \{x_1, \ldots x_n\}$ where the x_i represent mutually disjoint events.

- Then $P(X = x_i) + P(X = x_j)$ represents the probability of X being x_i *or* x_j.

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- Coarse-graining over all but one event gives *negation*: $\sum_{x_i \neq x_j} P(x_i) \ = \ 1 - P(X = x_j)$.

Effect algebras

Definition

An **effect algebra** $(E, \ovee, 0, 1)$ has

- *partial* commutative associative $\ovee$,
- with $a \ovee 0 = a$ for all a,
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Examples

- $[0, 1]$ with $a^\perp := 1 - a$.
- A Boolean algebra: addition defined when $a \wedge b = 0$ and then $a \ovee b = a \vee b$. $a^\perp$ is regular negation.
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- More generally $[0, 1]_V$ for any ordered vector space V.

Joint events

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Examples:

- $[0, 1]$.
- Any Boolean algebra: $a \ovee b := a \vee b$, $a \cdot b := a \wedge b$.
- $\{f : X \to [0, 1] \text{ continuous}\}$ for a compact Hausdorff space X (i.e. unit interval of commutative unital C*-algebra).

A categorical aside

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Let P be a poset and $a, b \in P$.

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  OMP morphism preserves orthogonality and $\vee$.

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Theorem (Jenča, 2015): $\mathbf{BPos}^K \cong \mathbf{EA}$

Theorem (Jacobs & Mandemaker 2012): Effect monoids are monoids in **EA**.

Countable sums

When we have infinite set of events $\{x_i\}_{i \in I}$, we want to be able to define union of countable events: $\sum_{j \in J} P(x_j)$.

Definition (informal)

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Equivalent definition

EA is ωEA iff increasing sequences $a_1 \leqslant a_2 \leqslant \ldots$ have a supremum.

Examples:

- $[0, 1]$.
- ω-complete Boolean algebra.
- $C(X, [0, 1])$ for X *basically disconnected*.

Our definition of abstract probabilities

So we want to model probabilities by an ω effect monoid $(M, 0, 1, \ovee, \perp, \cdot)$:

- It has partial sum $\ovee$.
- It has negation $\perp$ and min and max element 0 and 1.
- It has multiplication $\cdot$.
- It has suprema of increasing sequences.

Note: We are not requiring countable distributivity or commutativity of multiplication. This turns out to follow for free (non-trivially).

Characterising ω-effect-monoids

Theorem (Westerbaan, Westerbaan & vdW, 2020)

An ω-effect-monoid M embeds into $M_1 \oplus M_2$ where

- M_1 is an ω-complete Boolean algebra
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Corollary

ω-effect-monoids are commutative.

Call M *irreducible* when $M \cong M_1 \oplus M_2$ implies $M_i = \{0\}$.

Corollary

The only irreducible ω-effect-monoids are $\{0\}$, $\{0,1\}$ and $[0,1]$.

So why are probabilities modelled by $[0,1]$?
An answer: it is the only non-trivial irreducible ω-effect-monoid.

The result more category-theoretically

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Category of ω-effect-monoids is monadic over category of bounded posets.

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Theorem (Westerbaan^2 & vdW, 2020)

The only irreducible ω-effect-monoids are $\{0\}$, $\{0,1\}$ and $[0,1]$.

So: $[0,1]$ is unique non-initial, non-final irreducible Eilenberg-Moore algebra of particular monad over bounded posets.

Another way to phrase it

Theorem

There is a monad T over **BPos** such that $[0, 1]$ is the unique irreducible non-initial, non-final T-algebra.

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Theorem

There is a monad T over **BPos** such that $[0,1]$ is the unique irreducible non-initial, non-final T-algebra.
Furthermore, $\mathbf{BPos}^T \cong \omega\mathbf{EM}$ and these algebras have

- a partial order,
- a (partially defined) countable addition,
- a negation,
- and a multiplication.

So we have captured what is special about $[0,1]$ categorically.

Some things we can do with these results.

- A new Stone duality.
- (Characterise Generalised Probabilistic Theories).
- (Characterise *normal sequential* effect algebras)
- (Reconstruct quantum theory)

Directed-complete effect monoids

Definition

A subset $S \subseteq P$ of a poset P is *directed* when $\forall a, b \in S, \exists c \in S$ with $a \leqslant c$ and $b \leqslant c$.

P is *directed complete* when every directed subset has supremum.

Directed-complete effect monoids

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A subset $S \subseteq P$ of a poset P is *directed* when $\forall a, b \in S, \exists c \in S$ with $a \leqslant c$ and $b \leqslant c$.

P is *directed complete* when every directed subset has supremum.

Theorem (Westerbaan[2] & vdW, 2020)

A directed-complete effect monoid M is $M \cong M_1 \oplus M_2$ where

- M_1 is complete Boolean algebra.
- $M_2 := \{f : X \to [0,1] \text{ cont.}\}$ with X extremally disconnected.

Stone duality

Let **CBA** be category of complete Boolean algebras.
Recall that a space is Stonean when it is extremally disconnected compact Hausdorff.

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Definition

Let $\mathbf{Stone}_{\mathrm{sub}}$ be cat of Stonean spaces w/ designated clopen subset.
I.e. objects (X, A) where X is Stonean, and $A \subseteq X$ is clopen.
$f : (X, A) \to (Y, B)$ is $f : X \to Y$ continuous & $f(A) \subseteq B$.

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Theorem

Let **DCEM** be cat of directed-complete effect monoids.
Then $\mathbf{DCEM} \cong \mathbf{Stone}_{\mathrm{sub}}^{\mathrm{op}}$.

Summary

- Categorical characterisation of **Hilb** as nice †-category
- Characterisation of categories of form CPM(**C**).
- Operational characterisation of $\mathbf{CPM}_S$ for $S = \mathbb{C}$ or $S = \mathbb{R}$.
- Categorical characterisation of $[0, 1]$.

Open questions

- Characterise **fHilb** in similar way to **Hilb**.
- What are the possible phased rings in Tull-categories?
- Is there a clean categorical characterisation of **CPM**?
- And what about infinite-dimensional C*-algebras?

Thank you for your attention!

Chris Heunen & Andre Kornell 2021, arXiv:2109.07418
Axioms for the category of Hilbert spaces

Peter Selinger 2007, DOI:10.1016/j.entcs.2006.12.018
Dagger compact closed categories and completely positive maps

Bob Coecke 2007, DOI:10.1016/j.entcs.2008.04.014
Axiomatic description of mixed states from Selinger's CPM-construction

Sean Tull 2018, arXiv:1804.02265
A Categorical Reconstruction of Quantum Theory

Abraham Westerbaan, Bas Westerbaan & vdW 2019, arXiv:1912.10040
A characterisation of ordered abstract probabilities

vdW 2021, arXiv: 2106.10094
A Categorical Construction of the Real Unit Interval