



# Sequential Measurement characterises Quantum Theory

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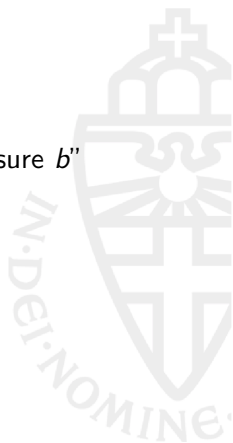
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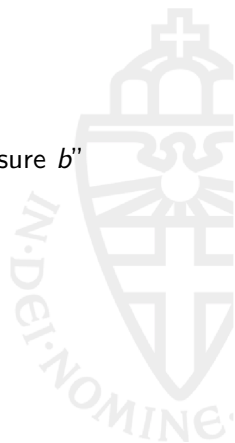
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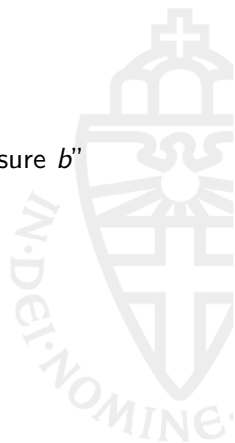




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In quantum theory  $a \& b := \sqrt{ab}\sqrt{a}$  and  $a | b$  when  $ab = ba$ .



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- But it can be recovered with one additional assumption.







# Generalised Probabilistic Theories

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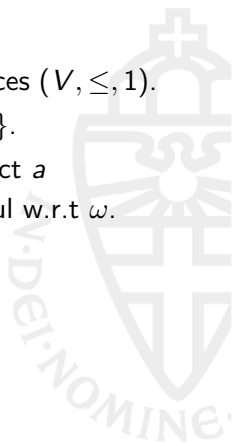
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$V$ 's with  $\omega(c) \leq 1 \forall \omega \implies c \leq 1$  are *order unit spaces*.



## Sequential Product: Additivity

- Start with an ensemble of systems. Measure  $a$  on all of them.

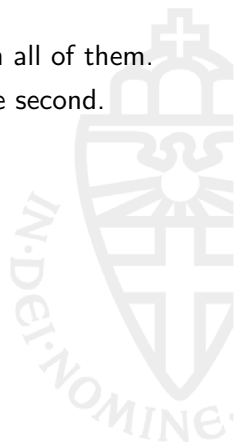






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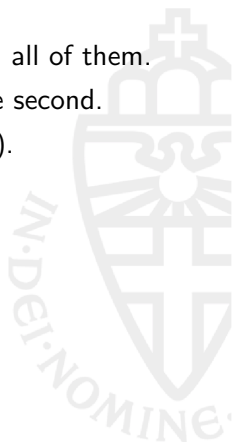
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Resistance to noise:  $a \mapsto a \& b$  is continuous.



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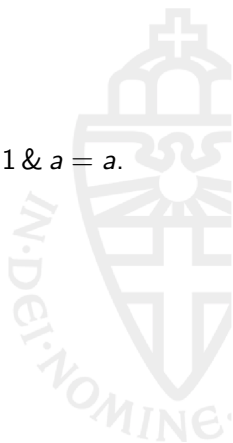
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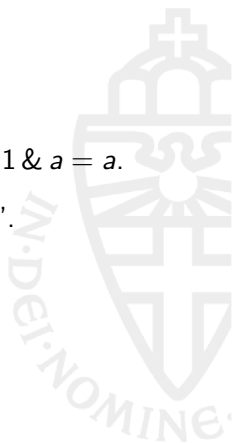
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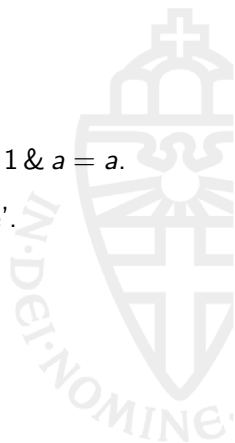
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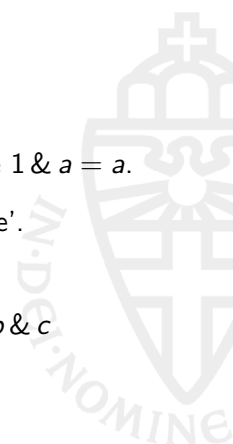
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- ⇒ Positive cone of  $V$  is homogeneous.



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## Koecher-Vinberg Theorem

An ordered vector space that is homogeneous and self-dual is a Euclidean Jordan algebra.





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Also true:

## Self-duality

A finite-dimensional sequential order space allows an inner product such that  $a \geq 0 \iff \langle a, b \rangle \geq 0$  for all  $b \geq 0$ .

Much harder to show! Requires recent characterisation of rank 2 homogeneous spaces (Ito, Lourenço 2017).

## Koecher-Vinberg Theorem

An ordered vector space that is homogeneous and self-dual is a Euclidean Jordan algebra.

Sequential effect spaces are just Jordan algebras in disguise!



## Composite systems

How do we get from EJAs to quantum theory?





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## Sequential Measurement Characterises Quantum Theory

A locally tomographic generalised probabilistic theory of sequential effect spaces embeds into **CStar**<sub>CP</sub>.



# Minimality of conditions

## The full theorem

A finite-dimensional order unit space  $V$  with  $E = \{a \in V ; 0 \leq a \leq 1\}$  and  $\& : E \times E \rightarrow E$  such that

- $a \& (b + c) = a \& b + a \& c$  and  $a \mapsto a \& b$  is continuous
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# Conclusion

Finite-dimensional order unit space  
Continuous sequential product

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Euclidean Jordan algebra

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⇒ **Sequential Measurement characterises Quantum Theory**







## Advertisements

Gudder & Greechie (2002): *Sequential Products on Effect Algebras*

vdW (2018): *Three Characterisations of the Sequential Product*  
arXiv:1803.08453

vdW (2018): *Sequential Measurement Characterises Quantum Theory*  
arXiv:1803.11139

Thank you for your attention