Quantum theory is a quasi-stochastic process theory

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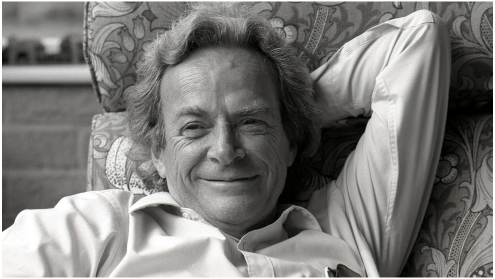

"The only difference between a probabilistic classical world and the equations of the quantum world is that somehow or other it appears as if the probabilities would have to go negative."

– Richard Feynman, 1981

A bit of background

- Wigner (1932): Representing a quantum state as a distribution over classical phase space allowing negative probabilities.

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- Negativity in representations is “equivalent” to contextuality (Spekkens 2008).
- Quantum speed up requires sufficient negativity in representations (Pashayan, Walman & Bartlett 2015).

Related work

- Appleby, Fuchs, Stacey, Zhu 2016 "Introducing the Qplex".
- Hardy 2013 "The duotensor framework"
- Ferrie & Emerson 2008 "Frame representations of quantum mechanics"

Informationally complete POVMs

Definition

- Let M_n be the set of $n \times n$ complex matrices.
- An *effect* is an $E \in M_n$ such that $0 \leq E \leq 1$.
- A *POVM* is a set of effects $\{E_i\}$ such that $\sum_i E_i = I_n$.

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Definition

A *quantum state* is $\rho \in M_n$ such that $\rho \geq 0$ and $\mathrm{tr}(\rho) = 1$.

Quantum states as probability distributions

Let $\rho \in M_n$ be a quantum state and $\{E_i\}$ a POVM.
$\rightarrow p(i) = \mathrm{tr}(\rho E_i)$ forms a probability distribution.

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Now:

$$p(i) = \text{tr}(\rho E_i) = \sum_j \alpha_j \, \text{tr}\left(\frac{E_j}{\text{tr}(E_j)} E_i\right)$$

Quantum states as probability distributions - cont.

$$p(i) = \operatorname{tr}(\rho E_i) = \sum_j \alpha_j \operatorname{tr}\left(\frac{E_j}{\operatorname{tr}(E_j)} E_i\right)$$

Define the *transition matrix* $T_{ij} = \operatorname{tr}\left(\frac{E_j}{\operatorname{tr}(E_j)} E_i\right)$.

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then we can succinctly write

$$p = T\alpha \qquad \text{or equivalently} \qquad \alpha = T^{-1} p$$

Which allows us to reconstruct the original state:

$$\rho = \sum_i (T^{-1} p)_i \frac{E_i}{\mathrm{tr}(E_i)}$$

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NOTE: T^{-1} can contain negative components!

(quasi-)stochasticity

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Definition

- A real-valued matrix S is called *stochastic* when $S_{ij} \in \mathbb{R}_{\geq 0}$ for all i, j and all the columns sum up to 1.
- It is *quasi-stochastic* when the positivity requirement is dropped.
- S is *doubly* (quasi-)stochastic when its transpose is also (quasi-)stochastic.

Stochastic matrices are precisely those matrices that map the space of probability distributions to itself.

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Stochastic matrices are precisely those matrices that map the space of probability distributions to itself.
S stochastic $\not\Rightarrow$ S^{-1} stochastic (when it exists).

Quantum channels as quasi-stochastic matrices

Let $\Phi : M_n \to M_m$ be a CPTP-map and fix MIC-POVMs $\{E_i\}$ and $\{E_j'\}$ on respectively M_n and M_m. Let T be the transition matrix for $\{E_i\}$.

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$$p(i) := \mathrm{tr}(\rho E_i) \quad \Rightarrow \quad \rho = \sum_i (T^{-1}p)_i \frac{E_i}{\mathrm{tr}(E_i)}$$

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Define

$$Q(\Phi)_{ij} = \mathrm{tr}\left(\Phi\left(\frac{E_j}{\mathrm{tr}(E_j)}\right) E'_i\right)$$

Then $q = Q(\Phi)T^{-1}p$

Now that we've got that out of the way...

...time for some new stuff

Quantum channels as quasi-stochastic matrices - cont.

$\Phi : M_n \to M_m$ and $\Psi : M_m \to M_k$ with MIC-POVMs $\{E_i\}$, $\{E_i'\}$ and $\{E_i''\}$, and transition matrices T, T' and T''.

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$$Q(\Phi)_{ij} = \text{tr}\left(\Phi\left(\frac{E_j}{\text{tr}(E_j)}\right)E_i'\right) \quad Q(\Psi)_{ij} = \text{tr}\left(\Psi\left(\frac{E_j'}{\text{tr}\left(E_j'\right)}\right)E_i''\right)$$

and set $\tau = (\Psi \circ \Phi)(\rho)$ with distribution $r(i) = \text{tr}(\tau E_i'')$.

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Quantum theory as a quasi-stochastic process theory

Definition

Let **CPTP** be the category with objects natural numbers and morphisms CPTP maps $\Phi : M_n \to M_m$.
Let **QStoch** be the category with objects natural numbers and morphisms quasi-stochastic matrices.
Note: Density matrices are equivalent to $\hat{\rho} : M_1 = \mathbb{C} \to M_n$.

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Fix $\forall n \in \mathbb{N}$ MIC-POVMs $\{E_i^{(n)}\}$ with transition matrices T_n.
Let F_E : **CPTP** $\to$ **QStoch** be a functor with $F_E(n) = n^2$ and $F_E(\Phi : M_n \to M_m) = Q(\Phi) T_n^{-1}$ where

$$Q(\Phi)_{ij} = \text{tr}\left(\Phi\left(\frac{E_j^{(n)}}{\text{tr}\left(E_j^{(n)}\right)} \right) E_i^{(m)} \right)$$

Properties of the quasi-stochastic representation

Theorem

F_E : $\mathbf{CPTP} \to \mathbf{QStoch}$ is indeed a functor. It preserves convex mixtures of channels and is faithful.

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A different set of MIC-POVMs gives a different functor, but:

Theorem

Any two functors $F_E, F_{E'}$: **CPTP** $\to$ **QStoch** arising from a choice of MIC-POVMS are naturally isomorphic.

Preservation of tensor product

Definition: Strong monoidal functors

A functor $F : \mathbb{A} \to \mathbb{B}$ is called *strong monoidal* if there exist isomorphisms $\alpha_{A,B}$ for every pair of objects A and B such that $\alpha_{B_1,B_2} \circ (F(f_1) \otimes F(f_2)) = F(f_1 \otimes f_2) \circ \alpha_{A_1,A_2}$ for all morphisms $f_i : A_i \to B_i$ satisfying some coherence conditions.

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Theorem

The functor F_E : **CPTP** $\to$ **QStoch** is strong monoidal.

NOTE: You need minimality of the POVMs for this!

Preservation of adjoints

Definition: Linear algebraic adjoint

Let $A : (V, \langle \cdot, \cdot \rangle) \to (W, \langle \cdot, \cdot \rangle)$ be a a linear map. It's *adjoint* is a map $A^\dagger : (W, \langle \cdot, \cdot \rangle) \to (V, \langle \cdot, \cdot \rangle)$ such that

$$\langle v, A^\dagger w \rangle = \langle Av, w \rangle$$

e.g. adjoint of real matrix is the transpose
and adjoint of $\hat{U}(A) = UAU^\dagger$ is $\hat{U}^\dagger(A) = U^\dagger AU$.

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The adjoint of a CPTP map is CPTP if and only if it is *unital*.

Question: Does F_E preserve the adjoint of unital channels?
Answer: No! (in general)

Symmetric Informationally Complete POVMs

Definition

A MIC-POVM $\{E_i\}$ is called *symmetric* when

$$\exists \alpha, \beta : \forall i, j : \mathrm{tr}(E_i E_j) = \alpha \delta_{ij} + \beta$$

NOTE: The usual definition requires all E_i to be rank 1.

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Theorem

The functor F_E : **CPTP** $\rightarrow$ **QStoch** preserves the adjoint of unital channels, e.g. $F(\Phi^\dagger) = F(\Phi)^\dagger$, if and only if all associated MIC-POVMS are symmetric.

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- Language of category theory is a good fit for talking about quasi-stochastic representations of quantum theory.
- Yet again a special role for symmetric IC-POVMs.
- Construction also applies to causal operational probabilistic theories.

- **QStoch** doesn't 'care' about positivity. Can this be fixed?
- Can we 'simulate' causal OPTs using quantum theory with these representations?

Thank you for your attention