Self-duality and Jordan structure of quantum theory follow from homogeneity and pure transitivity

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Three special properties of quantum theory

- Self-duality: We can map states into effects by an inner product.
- Pure transitivity: We can map any **pure** state to any other **pure** state by a reversible transformation.
- Homogeneity: We can map any **mixed** state to any other **mixed** state by a probabilistically reversible transformation.

We show that in any GPT:

Homogeneity + pure transitivity $\implies$ *self-duality.*

From this follows:

Homogeneity + pure transitivity + local tomography uniquely defines quantum theory

What is self-duality?

In a **Generalised probabilistic theory** (GPT) we describe a system by a

- convex **state space** Ω,
- convex **effect space** E,
- affine **probability function** $(\omega, e) \in [0, 1]$ for $\omega \in \Omega$ and $e \in E$.

In quantum theory:

- $\Omega \subseteq M_n(\mathbb{C})_{\text{sa}}$ are density matrices,
- $E \subseteq M_n(\mathbb{C})_{\text{sa}}$ are positive sub-unital matrices,
- $(\cdot, \cdot)$ given by inner product $\langle A, B \rangle := \text{tr}(AB)$.

Something peculiar: Ω and E belong to **same** space $M_n(\mathbb{C})_{\text{sa}}$,
and are related by inner product.
This is **self-duality**.

Self-duality

Definition (informal)

A system is **self-dual** when (unnormalized) states can be identified with the effects by a probability-determining inner product.

Self-duality is ‘rare’ amongst GPTs:

Koecher-Vinberg theorem

self-duality + *homogeneity* = Jordan algebra = ‘almost’ quantum.

GPTs as vector spaces

- We can embed state space Ω into vector space V.
- Then E is subset of $V^* := \{f : V \to \mathbb{R}\}$, by $e(\omega) := (\omega, e)$.
- In finite dimension, V is **always** linearly isomorphic to V^* (via a choice of basis), and this defines an inner product.

Q: So what exactly is special about self-duality?

A: General inner products don't map valid states to valid effects.

GPTs as ordered vector spaces

We were missing crucial information about the vector space:

- We can order V by $a \leq b$ iff $(a, e) \leq (b, e)$ for all $e \in E$.
- This gives a **positive cone** $V_+ := \{v \geq 0 \,|\, v \in V\}$.
- We have $\Omega \subseteq V_+$.
- (In QT, positive cone = {positive-semidefinite matrices}.)
- Can also order the dual V^*, to get $E \subseteq (V^*)_+$.
- Desired inner product should hence at least preserve positivity.

Self-dual inner product

Definition

Let V be an ordered vector space.
An inner product $\langle \cdot, \cdot \rangle$ on V is **self-dualising** when

$$\langle v, w \rangle \geq 0 \text{ for all } w \geq 0 \iff v \geq 0.$$

Equivalently: view $\langle \cdot, \cdot \rangle$ as $\Phi : V \to V^*$ by $\Phi(v)(w) = \langle v, w \rangle$.
Then $\langle \cdot, \cdot \rangle$ is self-dual iff Φ is an **order isomorphism**:

$$\Phi(v) \geq 0 \iff v \geq 0.$$

Note

The existence of just an order iso $\Phi : V \to V^*$ is known as **weak** self-duality. Weak SD is necessary for state-teleportation protocols in GPTs (Barnum *et al.* 2012).

Another useful property of quantum theory

Homogeneity: ‘the positive cone is maximally symmetric.‘

- In finite dimension, vector spaces have a canonical topology.
- This allows us to talk about the **interior** of the positive cone.
- In a GPT, a state ω is in the interior, if it is **completely mixed**: $(\omega, e) > 0$ for all $e \in E$.
- In QT, a density matrix ρ is in the interior iff it is full-rank iff it is invertible.

Definition

A cone V_+ is **homogeneous** when for any two interior states $v, w \in V_+$, there is an order iso Φ, such that $\Phi(v) = w$.

Homogeneity in quantum theory

Definition

A cone V_+ is **homogeneous** when for any two interior states $v, w \in V_+$, there is an order iso Φ, such that $\Phi(v) = w$.

In quantum theory:

- Let ρ and σ be full-rank (unnormalised) states in $M_n(\mathbb{C})_{sa}$.
- Define $\Phi(A) := \sqrt{\sigma}\sqrt{\rho^{-1}}A\sqrt{\rho^{-1}}\sqrt{\sigma}$.
- Φ is certainly positive. Can also easily construct a positive inverse.
- Hence Φ is an order iso.
- And we see that $\Phi(\rho) = \sigma$.

So quantum systems are homogeneous.

Homogeneity operationally

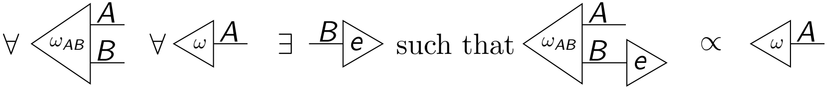
Mathematical meaning of homogeneity:
'*Group of order-symmetries acts transitively on the interior cone*'
or '*on an order-theoretic level, every internal point is equivalent*'

Q: What is the operational meaning?
An answer:

Theorem (based on Barnum *et al.* 2013)

If a system in a GPT is irreducible and allows **universal self-steering**, then it is homogeneous.

Informally, we say a system B **universally steers** A, if for every bipartite state ω_{AB} we can induce any* state on A by observing the right effect on B.



Self-duality and homogeneity

Koecher-Vinberg theorem

Let V be a homogeneous and self-dual ordered vector space.
Then V is order-isomorphic to a **Euclidean Jordan algebra** (EJA).

von Neumann, Wigner, Jordan classification

Any EJA is a direct sum of

- $M_n(\mathbb{C})_{sa}$: complex quantum systems
- $M_n(\mathbb{R})_{sa}$: real quantum systems
- $M_n(\mathbb{H})_{sa}$: quaternionic quantum systems
- $M_3(\mathbb{O})_{sa}$: a 3-dimensional octonionic system
- spin-factors: systems where Ω is an n-sphere (i.e. 'generalised qubits').

$\Rightarrow$ EJAs are 'almost-quantum' systems.

So what now?

- Koecher-Vinberg theorem is very powerful.
- Homogeneity has operational interpretation (steering).
- Self-duality does not.
- Can we replace it with some other nicer/operational property?

Pure transitivity

Definition

In a GPT system, a **pure state** is a convex extremal element of Ω:

$$\omega \in \Omega \text{ pure iff } \omega = \lambda\omega_1 + (1-\lambda)\omega_2 \implies \omega_1 = \omega_2$$

Definition

In a GPT, a **reversible** transformation is an affine $\Phi : \Omega \to \Omega$ which has an inverse Φ^{-1} such that $\Phi \circ \Phi^{-1} = \text{id} = \Phi^{-1} \circ \Phi$.

In QT:

- Pure states are what you think.
- Reversible transformations correspond to unitaries.

Definition

A GPT system satisfies **pure transitivity** iff for any pure states ω_1, ω_2 we can find a reversible transformation Φ such that $\Phi(\omega_1) = \omega_2$.

Operational interpretation of pure transitivity

Definition

A GPT system satisfies **pure transitivity** iff for any pure states ω_1, ω_2 we can find a reversible transformation Φ such that $\Phi(\omega_1) = \omega_2$.

Pure transitivity follows from *essential uniqueness of purification* + *pure conditioning* (that pure measurements preserve pure states).

More philosophically:

- If we consider pure states the 'real' states of the theory,
- and we consider reversible transformations as the 'real' dynamics,
- then failure of pure transitivity would mean two states of a system are not transformable into each other.
- But then isn't our definition of system is wrong?

Comparing homogeneity and pure transitivity

Recall $\Omega \subseteq V_+ \subseteq V$.

- $\Phi : V \to V$ is order iso when $\Phi(v) \geq 0 \iff v \geq 0$.
- It is a **normalised** order iso when $\Phi(\Omega) = \Omega$.
- Reversible transformations are normalised order iso's.

Pure transitivity

for all pure $\omega_1, \omega_2 \in \Omega$ there exists a normalised order iso Φ such that $\Phi(\omega_1) = \omega_2$.

Homogeneity

for all interior $v_1, v_2 \in V_+$ there exists an order iso Φ such that $\Phi(v_1) = v_2$.

Note: order iso's are rescaled **probabilistically reversible** transformations: $\Phi \circ \Phi^\sharp = p\,\mathrm{id}$

Our results

Theorem

Let V be a homogeneous ordered vector space that satisfies pure transitivity. Then V is self-dual.

Corollary (via Koecher-Vinberg theorem)

Such a vector space is then order-isomorphic to a Euclidean Jordan algebra.

Some more corollaries

Definition

We say Ω satisfies **continuous** pure transitivity when for all pure $\omega_1, \omega_2 \in \Omega$ there is a family Φ_t of reversible transformations for $t \in [0, 1]$ such that $t \mapsto \Phi_t(v_1)$ is a continuous path from v_1 to v_2.

Corollary

The state space of a system that satisfies continuous pure transitivity and universal self-steering is order-isomorphic to a Euclidean Jordan algebra.

Reconstructing quantum theory

We can reconstruct Jordan algebras. But can we restrict to just the quantum systems?

Definition

We say the composite of systems A and B is **locally tomographic** if $\dim V_{AB} = \dim V_A \cdot \dim V_B$ (i.e. when product states/effects span the composite state/effect space).

Theorem

Let A be a system in a GPT where composites are locally tomographic and every state space is homogeneous and satisfies continuous pure transitivity. Then $V_A \cong M_n(\mathbb{C})_{sa}$.

The proof

Theorem

Let V be a homogeneous ordered vector space that satisfies pure transitivity. Then V is self-dual.

- Vinberg (1963) showed that each homogeneous V has a non-zero subspace V^c, such that
- $V_+^c := V^c \cap V_+$ is homogeneous and **self-dual**.
- It turns out V^c is invariant under normalised order iso's of V.
- V^c has at least one pure state ω_c that is also a pure state of V.
- Now let ω in V be pure. With pure transitivity we find a normalised order iso Φ such that $\Phi(\omega_c) = \omega$.
- But $\Phi(V^c) = V^c$, so $\omega \in V^c$.
- Hence V and V^c have the same pure states.
- Hence $V^c = V$.

Conclusion

- Self-duality follows from homogeneity and pure transitivity.
- Homogeneity and pure transitivity have an operational interpretation, so this gives an operational variant of the Koecher-Vinberg theorem.
- Also requiring local tomography uniquely pinpoints quantum theory.
- Could've instead assumed a 'dynamical correspondence': a mapping from reversible transformations to observables.
- This then hence gives a reconstruction purely in terms of the symmetries of the pure and mixed states.

Thank you for your attention!

Barnum, Ududec, vdW 2023, arXiv:2306.00362
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