Dichotomy between deterministic and probabilistic models in countably additive effectus theory

Kenta Cho — National Institute of Informatics, Japan
Bas Westerbaan — University College London
John van de Wetering — Radboud University Nijmegen

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Generalized Probabilistic Theories

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They consist of

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- and the **scalars** $\{s : I \to I\}$ are the real unit interval $[0, 1]$

Special operations:

- **States** $\mathrm{St}(A) := \{\omega : I \to A\}$
- **Effects** $\mathrm{Eff}(A) := \{p : A \to I\}$
- $p \circ \omega$ is probability that p holds on state ω

From GPTs to effectuses

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Solution: allow more general sets of scalars $\{s : I \to I\}$.
Result: effectus theory.

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Definition

A **partial commutative monoid** (PCM) $(X, \ovee, 0)$ is a set X with a *partial* associative commutative operation $\ovee$ with unit 0.

$$(x \ovee y) \ovee z = x \ovee (y \ovee z) \qquad x \ovee y = y \ovee x \qquad x \ovee 0 = x$$

Write $x \perp y$ when $x \ovee y$ is defined.

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- Let **Cstar** be category of unital C*-algebras with positive subunital maps.
  Homsets $\mathbf{Cstar}(\mathfrak{A}, \mathfrak{B})$ are PCMs:
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Pfn and **Cstar** are examples of **PCM-enriched categories**:

$$(f \ovee g) \circ h = (f \circ h) \ovee (g \circ h) \qquad h \circ (f \ovee g) = (h \circ f) \ovee (h \circ g)$$

Effect algebras

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Examples

- $[0, 1]$ with $a^\perp := 1 - a$.
- A Boolean algebra: $a \perp b$ when $a \wedge b = 0$ and then $a \ovee b = a \vee b$. $a^\perp$ is the regular negation.
- $\mathbf{Cstar}(\mathbb{C}, \mathfrak{A}) \cong [0, 1]_{\mathfrak{A}}$ with $a^\perp := 1 - a$.

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- **Pfn**: $I = \{*\}$, $\mathbf{Pfn}(A, I) \cong \mathscr{P}(A)$.
- $\mathbf{Cstar}^{\mathrm{op}}$: $I = \mathbb{C}$, $\mathbf{Cstar}^{\mathrm{op}}(\mathfrak{A}, I) \cong [0, 1]_{\mathfrak{A}}$.

Relating effectus to GPTs

Let $\mathbf{C}$ be an effectus, and $A \in \mathbf{C}$.

- Effects $\mathrm{Eff}(A) := \mathbf{C}(A, I)$ form effect algebra.
- States $\mathrm{St}(A) := \mathbf{C}(I, A)$ form 'abstract convex set'.

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- What are they in an effectus?
- $\mathbf{Pfn}(I, I) \cong \{0, 1\}$ & $\mathbf{Cstar}^{\mathrm{op}}(I, I) \cong [0, 1]$.
- In general: $\mathbf{C}(I, I)$ is effect algebra.
- But also has a 'multiplication' given by composition $I \xrightarrow{s} I \xrightarrow{t} I$.

Scalars in effectus

Definition

An **effect monoid** $(M, \ovee, 0, 1, \cdot)$ is an effect algebra with associative distributive multiplication:

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Examples:

- $[0, 1]$.
- Any Boolean algebra: $a \ovee b := a \vee b$, $a \cdot b := a \wedge b$.
- $\{f : X \to [0, 1] \text{ continuous}\}$ for a compact Hausdorff space X (i.e. unit interval of commutative unital C*-algebra).

Effectus over an effect monoid

Let $\mathbf{C}$ be effectus, $M := \mathbf{C}(I, I)$ an effect monoid and $A \in \mathbf{C}$.

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- $\text{Eff}(A)$ has action of scalars via $s \cdot p := s \circ p$.
- An M-**effect module** E is an effect algebra with suitable left M-action $\cdot : M \times E \to E$.
- $\mathbf{EMod}_M^{\text{op}}$ is an effectus. $\text{Eff} \colon \mathbf{C} \to \mathbf{EMod}_M^{\text{op}}$ is a functor.

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- A **weight** M-**module** X is a PCM with M-action $\cdot : M \times X \to X$ and suitable *weight* function $|\cdot| : X \to M$.
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Any effect monoid is the set of scalars of some effectus
Corollary: There are some weird effectuses out there

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A σ-PCM is a PCM where some sums of countable sets are defined.

Examples:

- **Pfn**(A, B).
- Let **Wstar** be category of von Neumann algebras with normal positive subunital maps. Then **Wstar**$(\mathfrak{A}, \mathfrak{B})$ is σ-PCM.

In fact: **Pfn** and **Wstar** are σ-**PCM enriched**.

σ-effectus

Definition

σ-effectus is σ-PCM-enriched effectus with countable coproducts.

Examples:

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It turns out that σ-effectuses are way more well-behaved.

σ-effect monoids

Proposition

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Theorem (Westerbaan, Westerbaan & vdW, LICS'20)

An ω-directed-complete effect monoid M embeds into $M_1 \oplus M_2$
where M_1 is a ω-complete Boolean algebra and
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Corollary

Scalars in a σ-effectus are commutative.

Normalisation in σ-effectuses

Theorem

Let $\mathbf{C}$ be a σ-effectus with $M = \mathbf{C}(I, I)$.
The following are equivalent.

- States in $\mathbf{C}$ can be normalized.
- Non-zero scalars are epi.
- M has a 'division' operation.
- M has no zero divisors ($a \cdot b = 0 \implies a = 0$ or $b = 0$).
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Furthermore, if any and thus all these conditions hold then $M \cong \{0\}$, $M \cong \{0, 1\}$ or $M \cong [0, 1]$.

Dichotomy between deterministic and probabilistic models

Hence: σ-effectuses with normalization come in three types:

- $\mathbf{C}(I, I) \cong \{0\}$: only holds when $\mathbf{C}$ is equivalent to the trivial single-object category with a single morphism.

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When appropriate operational *separation properties* are satisfied, we can say even more about these latter two cases.

Separation properties

Definition

An effectus $\mathbf{C}$ has **state-separation** when $f \circ \omega = g \circ \omega$ for all states $\omega \in \mathrm{Eff}(A)$ implies $f = g$ for any $f, g : A \to B$.
(i.e. iff St: $\mathbf{C} \to \mathbf{WMod}_M$ faithful)

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Non-Example

Category of effect algebras $\mathbf{EA}^{\mathrm{op}} \cong \mathbf{EMod}^{\mathrm{op}}_{\{0,1\}}$.
Let $P(\mathscr{H}) \in \mathbf{EA}$ be the projections on a Hilbert space $\mathscr{H}$ with $\dim(\mathscr{H}) > 2$. Then Kochen-Specker theorem says $\mathrm{St}(P(\mathscr{H})) = \{0\}$.

Classical deterministic effectuses

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Let $\mathbf{C}$ be a σ-effectus with $\mathbf{C}(I, I) \cong \{0, 1\}$ and state-separation. Then there is a faithful morphism of σ-effectuses $F : \mathbf{C} \to \mathbf{Pfn}$, and $\mathrm{St}(A) \cong F(A)$ for all $A \in \mathbf{C}$.

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Hence, such effectuses are entirely classical.

Corollary

Nonclassical σ-effectuses w/ normalisation **must** have scalars $[0, 1]$.

Convex embeddings of effectuses

A similar sort of embedding holds in the probabilistic setting.

Definition (informal)

An **order unit space** (OUS) $(V, \leqslant, 1)$ is a real ordered vector space such that V has no infinitesimals and $[0,1]_V$ generates V.

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A **Banach** OUS is furthermore complete in its canonical norm.
A σ-OUS furthermore has countable directed suprema.

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Then there is a faithful morphism of σ-effectuses
$F : \mathbf{C} \to \sigma\mathbf{BOUS}^{\mathrm{op}}$, the category of Banach σ-OUSes.

Hence: Probabilistic effectuses embed into the well-studied land of real ordered vector spaces.

Summary

- A non-trivial σ-effectus with normalisation is either deterministic or probabilistic.

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- So we managed to go from an abstract categorical framework to concrete well-studied standard settings.

Future work

- Study general σ-effectuses (without normalization).
- Find inherent categorical characterisation of σ-effectuses.

Thank you for your attention

References

Kenta Cho, Bas Westerbaan & vdW 2020, arXiv:2003.10245
Dichotomy between deterministic and probabilistic models in countably additive effectus theory

Abraham Westerbaan, Bas Westerbaan & vdW 2019, arXiv:1912.10040
A characterisation of ordered abstract probabilities

Kenta Cho PhD Thesis 2019, arXiv:1910.12198
Effectuses in Categorical Quantum Foundations

Bas Westerbaan PhD Thesis 2018, arXiv:1803.01911
Dagger and Dilation in the category of Von Neumann Algebras