Purity in Euclidean Jordan algebras

Joint work with Bram and Bas Westerbaan
{bram,bas}@westerbaan.name

John van de Wetering
wetering@cs.ru.nl

Institute for Computing and Information Sciences
Radboud University Nijmegen

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Overview

- Pure maps
- Effectus theory, Corners and Filters
- Purity and Euclidean Jordan algebras

Pure maps in quantum theory

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$\implies$ Different definitions give different results

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Q: How do we generalise this to more general theories?

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- K. Cho, B. Jacobs, B. Westerbaan & A. Westerbaan (2015): *Introduction to effectus theory*.
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Important notions in effectus theory: *quotient* and *comprehension*.

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Define $\lceil q \rceil = \sum_i q_i$. $\lfloor q \rfloor = \sum_{i; \lambda_i = 1} q_i$.

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$\xi : \lceil q \rceil \mathcal{A} \lceil q \rceil \to \mathcal{A}$ by $\xi(p) = \sqrt{q} p \sqrt{q}$ is a filter
The projection $\pi : \mathcal{A} \to \lfloor q \rfloor \mathcal{A} \lfloor q \rfloor$ is a corner.

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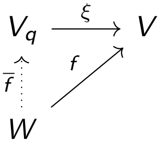

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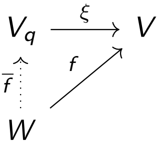

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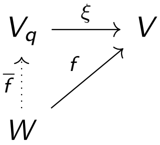

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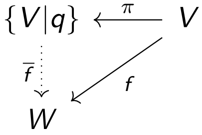

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- Is this even closed under composition?
  - $\Rightarrow$ In the right categories, yes.

Euclidean Jordan algebras

Definition

A *Euclidean Jordan algebra* (EJA) $(E, \langle \cdot, \cdot \rangle, *, 1)$ is a real Hilbert space with a product that satisfies:

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Example: $M_n(F)^{sa}$ — self-adjoint matrices over $F = \mathbb{R}, \mathbb{C}, \mathbb{H}$ with $A * B := \frac{1}{2}(AB + BA)$ and $\langle A, B\rangle := \mathrm{tr}(AB)$.

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Koecher-Vinberg Theorem

Every homogeneous and self-dual ordered vector space is an EJA.

As a result they crop up in a lot of places.

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Theorem

If a generalised probabilistic theory satisfies the above points, then the systems are EJAs.

see vdW (2018): *Reconstruction of quantum theory from universal filters*

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- Effectus definition of purity works on EJAs
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- Originally defined for von Neumann algebras.
  EJA+ vNA = JBW algebras. Does it work there?

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