Quantum Theory from First Principles
Lecture 3

John van de Wetering
john@vdwetering.name
http://vdwetering.name

Institute for Computing and Information Sciences
Radboud University Nijmegen

L'Agape Summer School
July 2020

Previously

- Introduced *generalised probabilistic theories* (GPTs) as a framework for studying alternative physical theories.
- They consist essentially of a *order unit spaces* V_A with *effects* $\mathrm{Eff}(A) \subseteq [0,1]_{V_A}$ and *states* $\mathrm{St}(A) \subseteq \mathrm{St}(V_A)$.
- Saw concepts like *coarse-graining*, *finite tomography* and *local tomography*.

Today

- Some modern reconstructions using the GPT framework.
- Reconstructing properties of quantum theory with ‘partial reconstructions’.
- Look at a reconstruction of my own based on *sequential measurement*.

Some modern reconstructions of quantum theory

Next time...

- Lucien Hardy, 2001:
  Quantum Theory From Five Reasonable Axioms.
- Chiribella, D'Ariano, Perinotti, 2011:
  Informational derivation of quantum theory.
- Masanes & Müller, 2011:
  A derivation of quantum theory from physical requirements.
- Barnum, Müller, Ududec, 2014: *Higher-order interference and single-system postulates characterizing quantum theory*

Hardy's five axioms

- Axiom 1: We are using the GPT framework.

Hardy's five axioms

- Axiom 1: We are using the GPT framework.
- Axiom 2: The tomographic dimension of a system is a function of the *informational dimension*, and takes the 'minimal value consistent with the axioms'.

Hardy's five axioms

- Axiom 1: We are using the GPT framework.
- Axiom 2: The tomographic dimension of a system is a function of the *informational dimension*, and takes the 'minimal value consistent with the axioms'.
- Axiom 3: 'A system whose state is constrained to belong to an M-dimensional subspace (i.e. have support on only M of a set of N possible distinguishable states) behaves like a system of dimension M.'

Hardy's five axioms

- Axiom 1: We are using the GPT framework.
- Axiom 2: The tomographic dimension of a system is a function of the *informational dimension*, and takes the 'minimal value consistent with the axioms'.
- Axiom 3: 'A system whose state is constrained to belong to an M-dimensional subspace (i.e. have support on only M of a set of N possible distinguishable states) behaves like a system of dimension M.'
- Axiom 4: Theory satisfies local tomography, and informational dimension 'multiplies' over composite systems.

Hardy's five axioms

- Axiom 1: We are using the GPT framework.
- Axiom 2: The tomographic dimension of a system is a function of the *informational dimension*, and takes the 'minimal value consistent with the axioms'.
- Axiom 3: 'A system whose state is constrained to belong to an M-dimensional subspace (i.e. have support on only M of a set of N possible distinguishable states) behaves like a system of dimension M.'
- Axiom 4: Theory satisfies local tomography, and informational dimension 'multiplies' over composite systems.
- Axiom 5: 'There exists a continuous reversible transformation on a system between any two pure states of that system.'

Perfectly distinguishable states

A collection of states $\omega_1, \ldots, \omega_k \in \mathsf{St}(A)$ is *perfectly distinguishable* if there exists a measurement $a_1, \ldots a_k$ such that $\omega_i(a_j) = \delta_{ij}$.

Perfectly distinguishable states

A collection of states $\omega_1, \ldots, \omega_k \in \mathrm{St}(A)$ is *perfectly distinguishable* if there exists a measurement $a_1, \ldots a_k$ such that $\omega_i(a_j) = \delta_{ij}$.

The *informational dimension* I_A of A is the maximal size of a set of perfectly distinguishable states. Tomographic dimension is $T_A := \dim(V_A)$.

Perfectly distinguishable states

A collection of states $\omega_1, \dots, \omega_k \in \mathrm{St}(A)$ is *perfectly distinguishable* if there exists a measurement $a_1, \dots a_k$ such that $\omega_i(a_j) = \delta_{ij}$.

The *informational dimension* I_A of A is the maximal size of a set of perfectly distinguishable states. Tomographic dimension is $T_A := \dim(V_A)$.

> *Axiom 2: $T_A = f(I_A)$ for some $f : \mathbb{N} \to \mathbb{N}$ and if there is multiple options for f we pick the smallest one.*

Perfectly distinguishable states

A collection of states $\omega_1, \ldots, \omega_k \in \mathrm{St}(A)$ is *perfectly distinguishable* if there exists a measurement $a_1, \ldots a_k$ such that $\omega_i(a_j) = \delta_{ij}$.

The *informational dimension* I_A of A is the maximal size of a set of perfectly distinguishable states. Tomographic dimension is $T_A := \dim(V_A)$.

> *Axiom 2: $T_A = f(I_A)$ for some $f : \mathbb{N} \to \mathbb{N}$ and if there is multiple options for f we pick the smallest one.*

Examples

For $V_A = \mathbb{R}^n$ we have $I_A = T_A = n$.
For $V_A = M_n(\mathbb{C})_{\mathrm{sa}}$ we have $I_A = n$ and $T_A = n^2$.
For $V_A = M_n(\mathbb{R})_{\mathrm{sa}}$ we have $I_A = n$, $T_A = n(n+1)/2$.

Axiom 3 and 4

Axiom 3: A system whose state is constrained to belong to an M-dimensional subspace behaves like a system of dimension M.

Axiom 3 and 4

Axiom 3: A system whose state is constrained to belong to an M-dimensional subspace behaves like a system of dimension M.

This is related to the existence of ‘filters’ for a system. We will get back to it later.

Axiom 3 and 4

> *Axiom 3: A system whose state is constrained to belong to an M-dimensional subspace behaves like a system of dimension M.*

This is related to the existence of 'filters' for a system. We will get back to it later.

> *Axiom 4: For all systems A and B we have* $T_{A\otimes B} = T_A T_B$ *and* $I_{A\otimes B} = I_A I_B$.

Pure states

Definition

Let C be a convex set. An element $a \in C$ is *extreme* when $a = pb + (1 - p)c$ for $0 < p < 1$ implies $b = c = a$.

Pure states

Definition

Let C be a convex set. An element $a \in C$ is *extreme* when $a = pb + (1-p)c$ for $0 < p < 1$ implies $b = c = a$.

We call the extreme points of $\mathrm{St}(A)$ *pure states*.

Pure states

Definition

Let C be a convex set. An element $a \in C$ is *extreme* when $a = pb + (1-p)c$ for $0 < p < 1$ implies $b = c = a$.

We call the extreme points of $\mathrm{St}(A)$ *pure states*.

Definition

A GPT satisfies *pure transitivity* if for each pair of pure states $\omega_1, \omega_2 \in \mathrm{St}(A)$ we can find a reversible transformation $\Phi : A \to A$ such that $\Phi(\omega_1) = \omega_2$.

Pure states

Definition

Let C be a convex set. An element $a \in C$ is *extreme* when $a = pb + (1-p)c$ for $0 < p < 1$ implies $b = c = a$.

We call the extreme points of $\mathrm{St}(A)$ *pure states*.

Definition

A GPT satisfies *pure transitivity* if for each pair of pure states $\omega_1, \omega_2 \in \mathrm{St}(A)$ we can find a reversible transformation $\Phi : A \to A$ such that $\Phi(\omega_1) = \omega_2$.
It satisfies *continuous transitivity* if we can find a family of reversible transformations Φ_t for $t \in [0,1]$ such that $\Phi_0(\omega_1) = \omega_1$ and $\Phi_1(\omega_1) = \omega_2$.

Pure states

Definition

Let C be a convex set. An element $a \in C$ is *extreme* when $a = pb + (1-p)c$ for $0 < p < 1$ implies $b = c = a$.

We call the extreme points of $\mathrm{St}(A)$ *pure states*.

Definition

A GPT satisfies *pure transitivity* if for each pair of pure states $\omega_1, \omega_2 \in \mathrm{St}(A)$ we can find a reversible transformation $\Phi : A \to A$ such that $\Phi(\omega_1) = \omega_2$.

It satisfies *continuous transitivity* if we can find a family of reversible transformations Φ_t for $t \in [0,1]$ such that $\Phi_0(\omega_1) = \omega_1$ and $\Phi_1(\omega_1) = \omega_2$.

Axiom 5: the GPT satisfies continuous transitivity.

Hardy's reconstruction restated

Theorem

Let $\mathbb{E}$ be a GPT where for all systems A and B

- $T_A = f(I_A)$ for some $f : \mathbb{N} \to \mathbb{N}$,
- $T_{A \otimes B} = T_A T_B$ and $I_{A \otimes B} = I_A I_B$,
- states with limited support act like they are on smaller systems.

If $\mathbb{E}$ additionally satisfies pure transitivity, then the GPT where f takes the smallest possible value is classical theory.
If $\mathbb{E}$ instead satisfies continuous transitivity, then the GPT where f takes the smallest possible value is quantum theory.

Chiribella, D'Ariano, Perinotti

Informational derivation of quantum theory (2011):

- Axiom 1: *causality*.

Chiribella, D'Ariano, Perinotti

Informational derivation of quantum theory (2011):

- Axiom 1: *causality*.
- Axiom 2: *Perfect distinguishability*: 'if a state is not completely mixed (i.e., if it cannot be obtained as a mixture from any other state), then there exists at least one state that can be perfectly distinguished from it.'

Chiribella, D'Ariano, Perinotti

Informational derivation of quantum theory (2011):

- Axiom 1: *causality*.
- Axiom 2: *Perfect distinguishability*: 'if a state is not completely mixed (i.e., if it cannot be obtained as a mixture from any other state), then there exists at least one state that can be perfectly distinguished from it.'
- Axiom 3: *Ideal compression*: 'every source of information can be encoded in a suitable physical system in a lossless and maximally efficient fashion.'

Chiribella, D'Ariano, Perinotti

Informational derivation of quantum theory (2011):

- Axiom 1: *causality*.
- Axiom 2: *Perfect distinguishability*: 'if a state is not completely mixed (i.e., if it cannot be obtained as a mixture from any other state), then there exists at least one state that can be perfectly distinguished from it.'
- Axiom 3: *Ideal compression*: 'every source of information can be encoded in a suitable physical system in a lossless and maximally efficient fashion.'
- Axiom 4: Local tomography.

Chiribella, D'Ariano, Perinotti

Informational derivation of quantum theory (2011):

- Axiom 1: *causality*.
- Axiom 2: *Perfect distinguishability*: 'if a state is not completely mixed (i.e., if it cannot be obtained as a mixture from any other state), then there exists at least one state that can be perfectly distinguished from it.'
- Axiom 3: *Ideal compression*: 'every source of information can be encoded in a suitable physical system in a lossless and maximally efficient fashion.'
- Axiom 4: Local tomography.
- Axiom 5: *Pure conditioning*: A local atomic effect applied to a pure composite state results in a pure state.

Chiribella, D'Ariano, Perinotti

Informational derivation of quantum theory (2011):

- Axiom 1: *causality*.
- Axiom 2: *Perfect distinguishability*: 'if a state is not completely mixed (i.e., if it cannot be obtained as a mixture from any other state), then there exists at least one state that can be perfectly distinguished from it.'
- Axiom 3: *Ideal compression*: 'every source of information can be encoded in a suitable physical system in a lossless and maximally efficient fashion.'
- Axiom 4: Local tomography.
- Axiom 5: *Pure conditioning*: A local atomic effect applied to a pure composite state results in a pure state.
- Axiom 6: Every state has an 'essentially unique' *purification*.

Refinements and faces

Definition

If $\omega = \sum_i p_i \omega_i$ for states $\omega, \omega_i \in \mathrm{St}(A)$ and probabilities $p_i \in [0, 1]$, we say each ω_i *refines* ω.

Refinements and faces

Definition

If $\omega = \sum_i p_i \omega_i$ for states $\omega, \omega_i \in \mathrm{St}(A)$ and probabilities $p_i \in [0, 1]$, we say each ω_i *refines* ω.
The *face identified by* ω is the convex subset $F_\omega \subseteq \mathrm{St}(A)$ consisting of refinements of ω.

Refinements and faces

Definition

If $\omega = \sum_i p_i \omega_i$ for states $\omega, \omega_i \in \mathrm{St}(A)$ and probabilities $p_i \in [0, 1]$, we say each ω_i *refines* ω.
The *face identified by* ω is the convex subset $F_\omega \subseteq \mathrm{St}(A)$ consisting of refinements of ω.
ω is *completely mixed* when every state $\omega' \in \mathrm{St}(A)$ refines ω, i.e. when $F_\omega = \mathrm{St}(A)$.

Refinements and faces

Definition

If $\omega = \sum_i p_i \omega_i$ for states $\omega, \omega_i \in \mathrm{St}(A)$ and probabilities $p_i \in [0,1]$, we say each ω_i *refines* ω.
The *face identified by* ω is the convex subset $F_\omega \subseteq \mathrm{St}(A)$ consisting of refinements of ω.
ω is *completely mixed* when every state $\omega' \in \mathrm{St}(A)$ refines ω, i.e. when $F_\omega = \mathrm{St}(A)$.

In any GPT it is always the case that a completely mixed state is not perfectly distinguishable from any other state.

Refinements and faces

Definition

If $\omega = \sum_i p_i \omega_i$ for states $\omega, \omega_i \in \mathrm{St}(A)$ and probabilities $p_i \in [0, 1]$, we say each ω_i *refines* ω.
The *face identified by* ω is the convex subset $F_\omega \subseteq \mathrm{St}(A)$ consisting of refinements of ω.
ω is *completely mixed* when every state $\omega' \in \mathrm{St}(A)$ refines ω, i.e. when $F_\omega = \mathrm{St}(A)$.

In any GPT it is always the case that a completely mixed state is not perfectly distinguishable from any other state.

Axiom 2: For $\omega \in St(A)$ not completely mixed, there exists $\sigma \in St(A)$ that is perfectly distinguishable from ω.

Refinements and faces

Definition

If $\omega = \sum_i p_i \omega_i$ for states $\omega, \omega_i \in \mathrm{St}(A)$ and probabilities $p_i \in [0, 1]$, we say each ω_i *refines* ω.
The *face identified by* ω is the convex subset $F_\omega \subseteq \mathrm{St}(A)$ consisting of refinements of ω.
ω is *completely mixed* when every state $\omega' \in \mathrm{St}(A)$ refines ω, i.e. when $F_\omega = \mathrm{St}(A)$.

In any GPT it is always the case that a completely mixed state is not perfectly distinguishable from any other state.

> *Axiom 2: For $\omega \in St(A)$ not completely mixed, there exists $\sigma \in St(A)$ that is perfectly distinguishable from ω.*

- Note 1: Axiom 2 follows from no-restriction hypothesis.

Refinements and faces

Definition

If $\omega = \sum_i p_i \omega_i$ for states $\omega, \omega_i \in \mathrm{St}(A)$ and probabilities $p_i \in [0,1]$, we say each ω_i *refines* ω.
The *face identified by* ω is the convex subset $F_\omega \subseteq \mathrm{St}(A)$ consisting of refinements of ω.
ω is *completely mixed* when every state $\omega' \in \mathrm{St}(A)$ refines ω, i.e. when $F_\omega = \mathrm{St}(A)$.

In any GPT it is always the case that a completely mixed state is not perfectly distinguishable from any other state.

Axiom 2: For $\omega \in St(A)$ not completely mixed, there exists $\sigma \in St(A)$ that is perfectly distinguishable from ω.

- Note 1: Axiom 2 follows from no-restriction hypothesis.
- Note 2: ω is pure iff $F_\omega = \{\omega\}$.

Ideal compression

- A *lossless compression* for a state $\omega \in \mathrm{St}(A)$ consists of an *encoding* $f : A \to B$ and *decoding* $g : B \to A$ such that $g(f(\omega_i)) = \omega_i$ when $\omega = \sum_i p_i \omega_i$.

Ideal compression

- A *lossless compression* for a state $\omega \in \mathrm{St}(A)$ consists of an *encoding* $f : A \to B$ and *decoding* $g : B \to A$ such that $g(f(\omega_i)) = \omega_i$ when $\omega = \sum_i p_i \omega_i$.
- I.e. $g \circ f$ is id when restricted to $F_\omega \subseteq \mathrm{St}(A)$.

Ideal compression

- A *lossless compression* for a state $\omega \in \mathrm{St}(A)$ consists of an *encoding* $f : A \to B$ and *decoding* $g : B \to A$ such that $g(f(\omega_i)) = \omega_i$ when $\omega = \sum_i p_i \omega_i$.
- I.e. $g \circ f$ is id when restricted to $F_\omega \subseteq \mathrm{St}(A)$.
- A compression for ω is *ideal* when B is 'as small as possible': when for each $\sigma \in \mathrm{St}(B)$ there is $\omega' \in F_\omega$ s.t. $f(\omega') = \sigma$.

Axiom 3: For every state there exists an ideal compression.

Atomic measurements and pure conditioning

For effects $a, a_i \in \mathrm{Eff}(A)$ we say each a_i *refines* a if $a = \sum_i a_i$.

Atomic measurements and pure conditioning

For effects $a, a_i \in \mathrm{Eff}(A)$ we say each a_i *refines* a if $a = \sum_i a_i$.
The refinement is *trivial* when $a_i = \lambda_i a$ for all i.
We call a *atomic* when it only has trivial refinements.

Atomic measurements and pure conditioning

For effects $a, a_i \in \text{Eff}(A)$ we say each a_i *refines* a if $a = \sum_i a_i$.
The refinement is *trivial* when $a_i = \lambda_i a$ for all i.
We call a *atomic* when it only has trivial refinements.

Axiom 5: Let $\omega \in St(A \otimes B)$ be pure and $a \in Eff(A)$ be atomic. Then $(a \otimes id) \circ \omega \in St(B)$ is a pure state.

Atomic measurements and pure conditioning

For effects $a, a_i \in \text{Eff}(A)$ we say each a_i *refines* a if $a = \sum_i a_i$.
The refinement is *trivial* when $a_i = \lambda_i a$ for all i.
We call a *atomic* when it only has trivial refinements.

> *Axiom 5: Let $\omega \in St(A \otimes B)$ be pure and $a \in Eff(A)$ be atomic. Then $(a \otimes id) \circ \omega \in St(B)$ is a pure state.*

Stronger version

> *Axiom 5': The composition of any two atomic transformations is again atomic.*

Purifications

Definition

A *purification* for $\omega \in \mathrm{St}(A)$ is a pure state $\sigma \in \mathrm{St}(A \otimes B)$ such that $\omega = (\mathrm{id} \otimes 1) \circ \sigma$, i.e. $\omega(a) = \sigma(a \otimes 1)$.

Purifications

Definition

A *purification* for $\omega \in \mathrm{St}(A)$ is a pure state $\sigma \in \mathrm{St}(A \otimes B)$ such that $\omega = (\mathrm{id} \otimes 1) \circ \sigma$, i.e. $\omega(a) = \sigma(a \otimes 1)$.

Axiom 6: Every state has a purification that is essentially unique: for two purifications $\sigma, \sigma' \in St(A \otimes B)$ of the same state there exists a reversible transformation $\Phi : B \to B$ such that $\sigma = (id \otimes \Phi) \circ \sigma'$.

Purifications

Definition

A *purification* for $\omega \in \mathrm{St}(A)$ is a pure state $\sigma \in \mathrm{St}(A \otimes B)$ such that $\omega = (\mathrm{id} \otimes 1) \circ \sigma$, i.e. $\omega(a) = \sigma(a \otimes 1)$.

> *Axiom 6: Every state has a purification that is essentially unique: for two purifications $\sigma, \sigma' \in St(A \otimes B)$ of the same state there exists a reversible transformation $\Phi : B \to B$ such that $\sigma = (id \otimes \Phi) \circ \sigma'$.*

Note: there is a trivial system I such that $A \otimes I \cong A$, which has a unique pure state id_I.

Purifications

Definition

A *purification* for $\omega \in \mathrm{St}(A)$ is a pure state $\sigma \in \mathrm{St}(A \otimes B)$ such that $\omega = (\mathrm{id} \otimes 1) \circ \sigma$, i.e. $\omega(a) = \sigma(a \otimes 1)$.

Axiom 6: Every state has a purification that is essentially unique: for two purifications $\sigma, \sigma' \in St(A \otimes B)$ of the same state there exists a reversible transformation $\Phi : B \to B$ such that $\sigma = (id \otimes \Phi) \circ \sigma'$.

Note: there is a trivial system I such that $A \otimes I \cong A$, which has a unique pure state id_I.
For any pure states $\omega, \omega' \in \mathrm{St}(A)$ both $\mathrm{id}_I \otimes \omega = \omega$ and $\mathrm{id}_I \otimes \omega' = \omega'$ are purifications for id_I.
So by essential uniqueness $\Phi(\omega') = \omega$ for some reversible $\Phi : A \to A$.

Purifications

Definition

A *purification* for $\omega \in \mathrm{St}(A)$ is a pure state $\sigma \in \mathrm{St}(A \otimes B)$ such that $\omega = (\mathrm{id} \otimes 1) \circ \sigma$, i.e. $\omega(a) = \sigma(a \otimes 1)$.

> *Axiom 6: Every state has a purification that is essentially unique: for two purifications $\sigma, \sigma' \in St(A \otimes B)$ of the same state there exists a reversible transformation $\Phi : B \to B$ such that $\sigma = (id \otimes \Phi) \circ \sigma'$.*

Note: there is a trivial system I such that $A \otimes I \cong A$, which has a unique pure state id_I.
For any pure states $\omega, \omega' \in \mathrm{St}(A)$ both $\mathrm{id}_I \otimes \omega = \omega$ and $\mathrm{id}_I \otimes \omega' = \omega'$ are purifications for id_I.
So by essential uniqueness $\Phi(\omega') = \omega$ for some reversible $\Phi : A \to A$.
$\Rightarrow$ get *pure transitivity* for free.

Informational derivation restated

Theorem

Let $\mathbb{E}$ be a GPT satisfying the 6 axioms below.

- Axiom 1: It is causal.
- Axiom 2: If a state cannot be perfectly distinguished from any other state, then it must be completely mixed.
- Axiom 3: Every state has an ideal compression.
- Axiom 4: It satisfies local tomography.
- Axiom 5: A composition of atomic processes is atomic.
- Axiom 6: Every state has an essentially unique purification.

Then $\mathrm{St}(A) \cong \mathrm{DO}(\mathbb{C}^n)$ for some n for each system A in $\mathbb{E}$.

Masanes & Müller reconstruction

A derivation of quantum theory from physical requirements, 2011:

- Axiom 1: ‘bitlike’ systems, that have informational dimension 2, are finite-dimensional.

Masanes & Müller reconstruction

A derivation of quantum theory from physical requirements, 2011:

- Axiom 1: ‘bitlike’ systems, that have informational dimension 2, are finite-dimensional.
- Axiom 2: local tomography.

Masanes & Müller reconstruction

A derivation of quantum theory from physical requirements, 2011:

- Axiom 1: ‘bitlike’ systems, that have informational dimension 2, are finite-dimensional.
- Axiom 2: local tomography.
- Axiom 3: systems and subspaces of equal informational dimension are isomorphic.

Masanes & Müller reconstruction

A derivation of quantum theory from physical requirements, 2011:

- Axiom 1: ‘bitlike’ systems, that have informational dimension 2, are finite-dimensional.
- Axiom 2: local tomography.
- Axiom 3: systems and subspaces of equal informational dimension are isomorphic.
- Axiom 4: Pure transitivity.

Masanes & Müller reconstruction

A derivation of quantum theory from physical requirements, 2011:

- Axiom 1: ‘bitlike’ systems, that have informational dimension 2, are finite-dimensional.
- Axiom 2: local tomography.
- Axiom 3: systems and subspaces of equal informational dimension are isomorphic.
- Axiom 4: Pure transitivity.
- Axiom 5: No-restriction hypothesis for bitlike systems.

Masanes & Müller reconstruction

A derivation of quantum theory from physical requirements, 2011:

- Axiom 1: 'bitlike' systems, that have informational dimension 2, are finite-dimensional.
- Axiom 2: local tomography.
- Axiom 3: systems and subspaces of equal informational dimension are isomorphic.
- Axiom 4: Pure transitivity.
- Axiom 5: No-restriction hypothesis for bitlike systems.

These axioms are only satisfied by classical theory, and quantum theory.

Commonalities

What do these three reconstructions have in common?

- They assume local tomography to control composites, and to remove real quantum theory from the possibilities.

Commonalities

What do these three reconstructions have in common?

- They assume local tomography to control composites, and to remove real quantum theory from the possibilities.
- They assume pure transitivity so they can define a unique ‘invariant’ state.

Commonalities

What do these three reconstructions have in common?

- They assume local tomography to control composites, and to remove real quantum theory from the possibilities.
- They assume pure transitivity so they can define a unique ‘invariant’ state.
- They have some kind of ‘filter’ axiom that allows them to reduce the problem to bitlike systems.

Commonalities

What do these three reconstructions have in common?

- They assume local tomography to control composites, and to remove real quantum theory from the possibilities.
- They assume pure transitivity so they can define a unique ‘invariant’ state.
- They have some kind of ‘filter’ axiom that allows them to reduce the problem to bitlike systems.

Let’s look at a quite different reconstruction.

Barnum, Müller, Ududec

Higher-order interference and single-system postulates characterizing quantum theory, 2014:

- Axiom 1: *spectrality*: any state can be written as a convex mixture of perfectly distinguishable pure states.

Barnum, Müller, Ududec

Higher-order interference and single-system postulates characterizing quantum theory, 2014:

- Axiom 1: *spectrality*: any state can be written as a convex mixture of perfectly distinguishable pure states.
- Axiom 2: pure 'frame' transitivity.

Barnum, Müller, Ududec

Higher-order interference and single-system postulates characterizing quantum theory, 2014:

- Axiom 1: *spectrality*: any state can be written as a convex mixture of perfectly distinguishable pure states.
- Axiom 2: pure 'frame' transitivity.
- Axiom 3: *No higher-order interference*: 'the interference pattern between mutually exclusive paths in an experiment is exactly the sum of the patterns which would be observed in all two-path sub-experiments, corrected for overlaps.'

Barnum, Müller, Ududec

Higher-order interference and single-system postulates characterizing quantum theory, 2014:

- Axiom 1: *spectrality*: any state can be written as a convex mixture of perfectly distinguishable pure states.
- Axiom 2: pure 'frame' transitivity.
- Axiom 3: *No higher-order interference*: 'the interference pattern between mutually exclusive paths in an experiment is exactly the sum of the patterns which would be observed in all two-path sub-experiments, corrected for overlaps.'
- Axiom 4: *Observability of energy*: 'there is non-trivial continuous reversible time evolution, and the generator of every such evolution can be associated to an observable ('energy') which is a conserved quantity.'

Frame transitivity

Definition

A *k-frame* is a set of pure states $\omega_1, \ldots, \omega_k \in \mathrm{St}(A)$ that are perfectly distinguishable.

> *Axiom 2: Given two k-frames* $\omega_1, \ldots, \omega_k$ *and* $\sigma_1, \ldots, \sigma_k$ *on* A *there is a reversible transformation* $\Phi : A \to A$ *such that* $\Phi(\omega_i) = \sigma_i$ *for all* i.

This is basically the strongest possible pure transitivity axiom.

Observability of energy

- Fix a system A. Denote its group of reversible transformations by G.

Observability of energy

- Fix a system A. Denote its group of reversible transformations by G.
- Under the assumptions of GPTs G is a compact Lie group.

Observability of energy

- Fix a system A. Denote its group of reversible transformations by G.
- Under the assumptions of GPTs G is a compact Lie group.
- Suppose its Lie algebra $\mathfrak{g}$ is not empty.

Observability of energy

- Fix a system A. Denote its group of reversible transformations by G.
- Under the assumptions of GPTs G is a compact Lie group.
- Suppose its Lie algebra $\mathfrak{g}$ is not empty.
- Each $X \in \mathfrak{g}$ generates a time evolution e^{tX} on $\mathrm{St}(A)$ and on V_A.

Observability of energy

- Fix a system A. Denote its group of reversible transformations by G.
- Under the assumptions of GPTs G is a compact Lie group.
- Suppose its Lie algebra $\mathfrak{g}$ is not empty.
- Each $X \in \mathfrak{g}$ generates a time evolution e^{tX} on $\mathrm{St}(A)$ and on V_A.
- An *energy observable assignment* is an injective linear map $\phi : \mathfrak{g} \to V_A$ such that $\phi(X)$ is conserved under e^{tX}, but not under all time evolutions.

Axiom 4: Every system has an energy observable assignment.

Relation to Jordan algebras

- Axioms 1-3 are satisfied only when each system A has V_A isomorphic to $\mathbb{R}^n$ or to a simple Euclidean Jordan algebra.

Relation to Jordan algebras

- Axioms 1-3 are satisfied only when each system A has V_A isomorphic to $\mathbb{R}^n$ or to a simple Euclidean Jordan algebra.
- Adding 'energy observability' only $M_n(\mathbb{C})_{\mathrm{sa}}$ remains.

Relation to Jordan algebras

- Axioms 1-3 are satisfied only when each system A has V_A isomorphic to $\mathbb{R}^n$ or to a simple Euclidean Jordan algebra.
- Adding 'energy observability' only $M_n(\mathbb{C})_{\mathrm{sa}}$ remains.
- Barnum & Hilgert in 2019 showed that Axiom 3 is actually superfluous: spectrality + frame transitivity = simple EJAs.

Enough full reconstructions, now lets do partial ones!

What is special about quantum theory?

What is special about quantum theory?

- Superposition of states!
- Entanglement!
- Wavefunction ‘collapse’!
- Heisenberg uncertainty!
- You can’t clone quantum states!
- You can calculate things faster!
- Bell nonlocality!

What is special about quantum theory?

- Superposition of states!
- Entanglement!
- Wavefunction 'collapse'!
- Heisenberg uncertainty!
- You can't clone quantum states!
- You can calculate things faster!
- Bell nonlocality!

...but are these special?

What is special about quantum theory?

- Superposition of states!
- Entanglement!
- Wavefunction 'collapse'!
- Heisenberg uncertainty!
- You can't clone quantum states!
- You can calculate things faster!
- Bell nonlocality!

...but are these special?

If the universe was governed by some other non-classical theory, would we not also see these properties?

Studying what is special

- Start with GPT framework.
- Formulate the property in GPT language.
- Assume some set of principles you like
  (But hopefully not enough to only get quantum theory!)
- If the property holds for all such GPTs then it is not special to quantum theory.

No-cloning/no-broadcasting

- A *cloning process* is some process $\Phi : A \to A \otimes A$ such that $\Phi(\omega) = \omega \otimes \omega$ for all $\omega \in \mathrm{St}(A)$.

No-cloning/no-broadcasting

- A *cloning process* is some process $\Phi : A \to A \otimes A$ such that $\Phi(\omega) = \omega \otimes \omega$ for all $\omega \in \mathsf{St}(A)$.
- Quantum theory famously has *no cloning*.

No-cloning/no-broadcasting

- A *cloning process* is some process $\Phi : A \to A \otimes A$ such that $\Phi(\omega) = \omega \otimes \omega$ for all $\omega \in \mathrm{St}(A)$.
- Quantum theory famously has *no cloning*.
- But: classically, there is also no process that can clone arbitrary mixed states.

No-cloning/no-broadcasting

- A *cloning process* is some process $\Phi : A \to A \otimes A$ such that $\Phi(\omega) = \omega \otimes \omega$ for all $\omega \in \mathrm{St}(A)$.
- Quantum theory famously has *no cloning*.
- But: classically, there is also no process that can clone arbitrary mixed states.
- A *Broadcasting process* is $\Phi : A \to A \otimes A$ satisfying $\Phi(\omega)(a \otimes 1) = \omega(a)$ and $\Phi(\omega)(1 \otimes a) = \omega(a)$. I.e. the marginal on each subsystem *acts* like a copy of ω.

No-cloning/no-broadcasting

- A *cloning process* is some process $\Phi : A \to A \otimes A$ such that $\Phi(\omega) = \omega \otimes \omega$ for all $\omega \in \mathrm{St}(A)$.
- Quantum theory famously has *no cloning*.
- But: classically, there is also no process that can clone arbitrary mixed states.
- A *Broadcasting process* is $\Phi : A \to A \otimes A$ satisfying $\Phi(\omega)(a \otimes 1) = \omega(a)$ and $\Phi(\omega)(1 \otimes a) = \omega(a)$. I.e. the marginal on each subsystem *acts* like a copy of ω.
- Classical theory has a broadcasting process, but quantum theory does not.

No-cloning/no-broadcasting

- A *cloning process* is some process $\Phi : A \to A \otimes A$ such that $\Phi(\omega) = \omega \otimes \omega$ for all $\omega \in \mathrm{St}(A)$.
- Quantum theory famously has *no cloning*.
- But: classically, there is also no process that can clone arbitrary mixed states.
- A *Broadcasting process* is $\Phi : A \to A \otimes A$ satisfying $\Phi(\omega)(a \otimes 1) = \omega(a)$ and $\Phi(\omega)(1 \otimes a) = \omega(a)$. I.e. the marginal on each subsystem *acts* like a copy of ω.
- Classical theory has a broadcasting process, but quantum theory does not.

Theorem (Barnum et al., 2007)

A causal GPT with local tomography has a broadcasting process for a system iff the system is classical.

Other classical properties

Some other properties that only hold if the system is classical, and not for any other GPTs: (see Barrett, *Information processing in generalized probabilistic theories*, 2007)

- Each state being a unique mixture of pure states.

Other classical properties

Some other properties that only hold if the system is classical, and not for any other GPTs: (see Barrett, *Information processing in generalized probabilistic theories*, 2007)

- Each state being a unique mixture of pure states.
- The existence of 'non-disturbing transformations'.

Other classical properties

Some other properties that only hold if the system is classical, and not for any other GPTs: (see Barrett, *Information processing in generalized probabilistic theories*, 2007)

- Each state being a unique mixture of pure states.
- The existence of ‘non-disturbing transformations’.
- A cloning process that can clone all pure states.

Other classical properties

Some other properties that only hold if the system is classical, and not for any other GPTs: (see Barrett, *Information processing in generalized probabilistic theories*, 2007)

- Each state being a unique mixture of pure states.
- The existence of 'non-disturbing transformations'.
- A cloning process that can clone all pure states.
- The existence of a measurement that can perfectly distinguish all pure states.

Theories with purification

Chiribella, D'Ariano, Perinotti, 2010

Any causal GPT with local tomography and essential uniqueness of purification has the following properties:

- It is not classical (hence does not have any of the properties of previous slide)
- Existence of 'maximally entangled' pure states.
- Possibility of probabilistic state teleportation.
- Every process can be dilated to reversible transformation.
- No bit commitment and no programming.

Phase groups

- Fix a measurement $a_1, \ldots, a_n \in \mathrm{Eff}(A)$ that perfectly distinguishes a maximal set of states.

Phase groups

- Fix a measurement $a_1, \ldots, a_n \in \mathrm{Eff}(A)$ that perfectly distinguishes a maximal set of states.
- Its *phase-group* $\mathcal{P}_{(a_i)}$ consists of the reversible transformations $\Phi : A \to A$ such that $\omega(\Phi(a_i)) = \omega(a_i)$ for all $\omega \in \mathrm{St}(A)$.

Phase groups

- Fix a measurement $a_1, \ldots, a_n \in \mathrm{Eff}(A)$ that perfectly distinguishes a maximal set of states.
- Its *phase-group* $\mathcal{P}_{(a_i)}$ consists of the reversible transformations $\Phi : A \to A$ such that $\omega(\Phi(a_i)) = \omega(a_i)$ for all $\omega \in \mathrm{St}(A)$.
- In quantum theory, for $a_i = |i\rangle\langle i|$ we have $\mathcal{P}_{(a_i)} = \{\mathrm{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n}) \; ; \; \theta_j \in [0, 2\pi]\} \cong U(1)^n$.

Phase groups

- Fix a measurement $a_1, \dots, a_n \in \mathrm{Eff}(A)$ that perfectly distinguishes a maximal set of states.
- Its *phase-group* $\mathcal{P}_{(a_i)}$ consists of the reversible transformations $\Phi : A \to A$ such that $\omega(\Phi(a_i)) = \omega(a_i)$ for all $\omega \in \mathrm{St}(A)$.
- In quantum theory, for $a_i = |i\rangle\langle i|$ we have $\mathcal{P}_{(a_i)} = \{\mathrm{diag}(e^{i\theta_1}, \dots, e^{i\theta_n}) \ ; \ \theta_j \in [0, 2\pi]\} \cong U(1)^n$.

Theorem (Garner et al., 2013)

A GPT has non-trivial phase-groups iff it is non-classical.

Power of correlations

Imagine the following *CHSH game* (proposed in 1969):

- Alice and Bob can communicate arbitrarily before the game starts, and share some bipartite state.

Power of correlations

Imagine the following *CHSH game* (proposed in 1969):

- Alice and Bob can communicate arbitrarily before the game starts, and share some bipartite state.
- Eve gives both Alice and Bob uniformly random bits s and t.

Power of correlations

Imagine the following *CHSH game* (proposed in 1969):

- Alice and Bob can communicate arbitrarily before the game starts, and share some bipartite state.
- Eve gives both Alice and Bob uniformly random bits s and t.
- Alice must reply with some bit a and Bob with some bit b.

Power of correlations

Imagine the following *CHSH game* (proposed in 1969):

- Alice and Bob can communicate arbitrarily before the game starts, and share some bipartite state.
- Eve gives both Alice and Bob uniformly random bits s and t.
- Alice must reply with some bit a and Bob with some bit b.
- Alice and Bob win if $a \oplus b = s \cdot t$.

Power of correlations

Imagine the following *CHSH game* (proposed in 1969):

- Alice and Bob can communicate arbitrarily before the game starts, and share some bipartite state.
- Eve gives both Alice and Bob uniformly random bits s and t.
- Alice must reply with some bit a and Bob with some bit b.
- Alice and Bob win if $a \oplus b = s \cdot t$.

What is the maximal probability that Alice and Bob can win?

Power of correlations

Imagine the following *CHSH game* (proposed in 1969):

- Alice and Bob can communicate arbitrarily before the game starts, and share some bipartite state.
- Eve gives both Alice and Bob uniformly random bits s and t.
- Alice must reply with some bit a and Bob with some bit b.
- Alice and Bob win if $a \oplus b = s \cdot t$.

What is the maximal probability that Alice and Bob can win?

- If they can share a classical state: $3/4 = 0.75$.

Power of correlations

Imagine the following *CHSH game* (proposed in 1969):

- Alice and Bob can communicate arbitrarily before the game starts, and share some bipartite state.
- Eve gives both Alice and Bob uniformly random bits s and t.
- Alice must reply with some bit a and Bob with some bit b.
- Alice and Bob win if $a \oplus b = s \cdot t$.

What is the maximal probability that Alice and Bob can win?

- If they can share a classical state: $3/4 = 0.75$.
- If they can share an arbitrary *non-signalling* box: $4/4 = 1$.

Power of correlations

Imagine the following *CHSH game* (proposed in 1969):

- Alice and Bob can communicate arbitrarily before the game starts, and share some bipartite state.
- Eve gives both Alice and Bob uniformly random bits s and t.
- Alice must reply with some bit a and Bob with some bit b.
- Alice and Bob win if $a \oplus b = s \cdot t$.

What is the maximal probability that Alice and Bob can win?

- If they can share a classical state: $3/4 = 0.75$.
- If they can share an arbitrary *non-signalling* box: $4/4 = 1$.
- If they can share a quantum state: $(2 + \sqrt{2})/4 \approx 0.85$

Power of correlations

Imagine the following *CHSH game* (proposed in 1969):

- Alice and Bob can communicate arbitrarily before the game starts, and share some bipartite state.
- Eve gives both Alice and Bob uniformly random bits s and t.
- Alice must reply with some bit a and Bob with some bit b.
- Alice and Bob win if $a \oplus b = s \cdot t$.

What is the maximal probability that Alice and Bob can win?

- If they can share a classical state: $3/4 = 0.75$.
- If they can share an arbitrary *non-signalling* box: $4/4 = 1$.
- If they can share a quantum state: $(2 + \sqrt{2})/4 \approx 0.85$

$2 + \sqrt{2}$ is known as the *Tsirelson bound*.
Why are quantum mechanical correlations strictly weaker than arbitrary non-signalling boxes?

Information causality

Principle of *information causality* (Pawłowski et al. 2009):

> *'The information gain that Bob can reach about a previously unknown to him data set of Alice, by using all his local resources and m classical bits communicated by Alice, is at most m bits."*

Information causality

Principle of *information causality* (Pawłowski et al. 2009):

> *'The information gain that Bob can reach about a previously unknown to him data set of Alice, by using all his local resources and m classical bits communicated by Alice, is at most m bits."*

(actual definition uses concept of *mutual information*)
Note that for $m = 0$ this is just no-signalling.

Information causality

Principle of *information causality* (Pawłowski et al. 2009):

> *'The information gain that Bob can reach about a previously unknown to him data set of Alice, by using all his local resources and m classical bits communicated by Alice, is at most m bits."*

(actual definition uses concept of *mutual information*)
Note that for $m = 0$ this is just no-signalling.

Theorem

If Alice and Bob can only share states that satisfy information causality, then they can win CHSH game with maximal probability $(2 + \sqrt{2})/4$.

Computational power

It is expected that **BQP**, the class of problems efficiently solvable by a quantum computer, is greater than **BPP**, problems solvable by a probabilistic classical computer.

Computational power

It is expected that **BQP**, the class of problems efficiently solvable by a quantum computer, is greater than **BPP**, problems solvable by a probabilistic classical computer.

The best-known bound for **BQP** is a class called **AWPP** (which contains for instance *graph isomorphism*).

Computational power

It is expected that **BQP**, the class of problems efficiently solvable by a quantum computer, is greater than **BPP**, problems solvable by a probabilistic classical computer.

The best-known bound for **BQP** is a class called **AWPP** (which contains for instance *graph isomorphism*).

Theorem (Lee & Barrett, 2015)

The computational power of any GPT satisfying local tomography is bounded by **AWPP**.

In 2019 it was showed that there is a GPT that reaches this bound.

So what is special about quantum theory?

- Superposition of states (nope, related to phase groups)
- Entanglement (nope)
- Heisenberg uncertainty (not really)
- You can't clone quantum states (nope)
- You can calculate things faster (not likely)
- Bell nonlocality (probably not)

Summary

- Saw a few different reconstructions using GPT framework.
- Saw that many qualitative properties of quantum theory are general properties of any non-classical theory.