Quantum Theory from First Principles
Lecture 1

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Why Quantum Theory?

Why Relativity?

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- Incredibly, his theory still seems correct for large scale structures.

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- Aesthetically pleasing.
  (reduces 'why relativity?' to 'why these principles?')
- Helps the search for generalisations
  (because you know you need to break one of these principles)

Back to quantum theory

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- 1925: Heisenberg, Born and Jordan developed matrix mechanics, Schrödinger developed wave mechanics.
- 1932: von Neumann, *Mathematische Grundlagen der Quantenmechanik*.

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- If the energy of a system is given by the observable H, then the system evolves as $|\psi(t)\rangle = e^{-itH} |\psi\rangle$.
- The Hilbert space of a composite system is given by the tensor product of the component Hilbert spaces.

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- First a lot of work was done in *quantum logic* (1960-1980), which was capped of by Sòler's theorem in 1995.
- Modern work (2000-2020) focuses more on *operational frameworks*.

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- Next lecture: order unit spaces, generalised probabilistic theories, modern reconstructions.
- Third lecture: ‘partial reconstructions’, and a reconstruction of my own.
- Final lecture: Generalising probabilities.

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The goal: Not to establish ‘the’ principles of quantum theory, but to convince you that such principles probably do exist.

Some classical results

Wigner's theorem

Question to be answered: Why is time evolution unitary?

Definition

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The *projective Hilbert space* is $P_1(\mathscr{H}) := \{\mathbb{C}\,|\psi\rangle \ ; \ |\psi\rangle \in \mathscr{H}\}$.
A *symmetry* of $P_1(\mathscr{H})$ is a bijective function $\Phi : P_1(\mathscr{H}) \to P_1(\mathscr{H})$ such that $T(\Phi\,|\psi\rangle, \Phi\,|\phi\rangle) = T(|\psi\rangle, |\phi\rangle)$ for all normalised representatives $|\psi\rangle, |\phi\rangle \in P_1(\mathscr{H})$.

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Wigner's theorem (1931)

For all symmetries Φ of $P_1(\mathscr{H})$ there exists an (anti-)unitary map $U : \mathscr{H} \to \mathscr{H}$ so that $\Phi(|\psi\rangle) = U\,|\psi\rangle$.

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- Evolving over a time t and then a time s, is the same as evolving over $t + s$, so
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- We furthermore expect $\lim_{t\to 0} U_t |\psi\rangle = |\psi\rangle$.
- It is then a *strongly continuous* one-parameter unitary group.

Stone's theorem on one-parameter unitary groups (1930)

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- Hence they are simultaneously diagonalisable: $U_t = V D_t V^\dagger$, with $D_t = \mathrm{diag}(e^{i\alpha_1(t)}, \dots)$ for some $\alpha_j : \mathbb{R} \to \mathbb{R}$.

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- By continuity α_j must be linear, and hence $\alpha_j(t) = h_j t$ for some $h_j \in \mathbb{R}$.
- Hence $D_t = e^{it\,\mathrm{diag}(h_1,\ldots)}$, and $U_t = e^{itH}$ for $H = V\,\mathrm{diag}(h_1, \ldots)V^\dagger$.

Wigner's theorem and Stone's theorem basically answer the question of why we get the Schrödinger equation: Given the structure of the states and transition probabilities, it is pretty much the only possible time dynamics you can have.

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Next stop is *Gleason's theorem*, that gives us the Born rule.

Density operators

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- In finite-dimension the mixed states are precisely the *density operators*: $\rho : \mathscr{H} \to \mathscr{H}$ positive with $\mathrm{tr}(\rho) = 1$.

Projective measurements

- Recall that observables are self-adjoint $A : \mathscr{H} \to \mathscr{H}$ and that their expectation value on $|\psi\rangle$ is $\langle\psi| A |\psi\rangle$.
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- The probability that we observe the outcome i on the state $|\psi\rangle$ is then $\langle\psi| P_i |\psi\rangle = \mathrm{tr}(P_i |\psi\rangle\langle\psi|)$.
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Definition

A *frame function* is a function $f : P(\mathscr{H}) \to [0,1]$ such that $\sum_i f(P_i) = 1$ whenever $\sum_i P_i = \mathrm{id}$ on $\mathscr{H}$.

Gleason's theorem

Gleason's theorem (1957)

Let $\mathscr{H}$ be a complex Hilbert space of dimension greater than 2, and let $f : P(\mathscr{H}) \to [0,1]$ be a frame function. Then there exists a density operator $\rho : \mathscr{H} \to \mathscr{H}$ such that $f(P) = \mathrm{tr}(P\rho)$ for all $P \in P(\mathscr{H})$.

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Some notes:

- The result does not hold when $\dim \mathscr{H} = 2$.
- The proof is notoriously difficult.
- However, it becomes almost trivial when we consider all *effects* instead of just projections.

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- With Stone's theorem we then get that any unitary evolution is implemented by some self-adjoint operator via the Schrödinger equation.

Combining Gleason, Wigner and Stone

- Suppose we know the observables correspond to self-adjoint operators on a complex Hilbert space.
- Then Gleason's theorem says the states are density operators, and that we have the Born rule.
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Hence someone infinitely smart could derive quantum mechanics just from knowing what the observables are.

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- Why are observables modelled by self-adjoint operators on a complex Hilbert space?
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- Why are composite systems described by a tensor product of Hilbert spaces?

The first of these questions is the hardest to answer, and is what we will spend most of this course on.

Remainder of this lecture

Three abstract mathematical structures that reduce to ‘quantum mechanics’ with suitable assumptions.

- C*-algebras
- Jordan algebras
- Orthomodular lattices

Operator algebras

Space of observables

Definition

Let $A : \mathscr{H} \to \mathscr{H}$. We call A *bounded* when there is some $c \in \mathbb{R}$ such that $\|Av\| \leqslant c\|v\|$ for all $v \in \mathscr{H}$. Denote space of bounded operators by $B(\mathscr{H})$. Define the *operator norm* $\|A\| = \inf_{v \neq 0} \|Av\|/\|v\|$.

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- The adjoint is a sesquilinear involution and satisfies $(AB)^\dagger = B^\dagger A^\dagger$ and $\left\|A^\dagger A\right\| = \|A\|\left\|A^\dagger\right\|$.

Abstracting the space of observables

Definition

A *C^*-algebra* $(\mathfrak{A}, \|\cdot\|, (\cdot)^*, \cdot)$ is a complex Banach space, equipped with a sesquilinear involution $(\cdot)^*$, that is simultaneously an associative algebra under $\cdot$ such that:

- $(a \cdot b)^* = b^* \cdot a^*$.
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Let $\mathfrak{A}$ be a C*-algebra. Then there exists a complex Hilbert space $\mathscr{H}$ and a norm-preserving, algebra-preserving, *-preserving embedding $\pi : \mathfrak{A} \to B(\mathscr{H})$.

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Hence, any 'abstract' C*-algebra embeds as a 'concrete' one on a Hilbert space.

Finite-dimensional C*-algebras

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Theorem

Let $\mathfrak{A}$ be a finite-dimensional C*-algebra. Then $\mathfrak{A} = M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$ for some $n_j \in \mathbb{N}$.

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- Note 3: They *are* closed under squares $A \in B(\mathscr{H})_{\mathrm{sa}} \implies A^2 \in B(\mathscr{H})_{\mathrm{sa}}$.
- So define $A * B := \frac{1}{2}[(A+B)^2 - A^2 - B^2] = \frac{1}{2}(AB + BA)$.

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then $(A * B) * C = 0$ while $A * (B * C) = \frac{1}{4} C$.
But, for all A and B:

$$A * (B * A^2) = (A * B) * A^2 \qquad \text{note } A^2 = A * A$$

This is the *Jordan identity*.

Jordan algebras

Definition

A *Jordan algebra* $(E, *)$ is a vector space with a bilinear commutative operation $*$ that satisfies the Jordan identity

$$a * (b * (a * a)) = (a * b) * (a * a) \text{ for all } a, b \in E.$$

Example

Let $(\mathfrak{A}, \cdot)$ be any associative algebra. Define $a * b := \frac{1}{2}(a \cdot b + b \cdot a)$.
Then $(\mathfrak{A}, *)$ is a Jordan algebra.
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In particular $(B(H), *)$ is a complex Jordan algebra.
But our goal was to have an algebra of self-adjoint operators.

Formally real Jordan algebras

Definition

A Jordan algebra is *formally real* when $\sum_j a_j^2 = 0$ implies $a_j = 0$ for all j.

Example

Let $\mathscr{H}$ be a real, complex or *quaternionic* Hilbert space. Then $(B(\mathscr{H})_{\mathrm{sa}}, *)$ is a formally real Jordan algebra.

Example

Let $\mathbb{O}$ denote the division algebra of the *octonions*. Then $(B(\mathbb{O}^3)_{\mathrm{sa}}, *) = (M_3(\mathbb{O})_{\mathrm{sa}}, *)$ is a formally real Jordan algebra.

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$(v, 0) * (w, 0) = (0, \langle v, w \rangle)$,
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Definition

The *n-dimensional spin factor* S_n is defined as $S_n = (\mathscr{H} \oplus \mathbb{R}, *)$ where $\mathscr{H} = \mathbb{R}^n$ is the unique n-dimensional real Hilbert space, and $*$ is defined as $(v, s) * (w, t) = (tv + sw, \langle v, w \rangle + st)$.
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$S_2 \cong M_2(\mathbb{R})_{\mathrm{sa}}$, $S_3 \cong M_2(\mathbb{C})_{\mathrm{sa}}$, $S_5 \cong M_2(\mathbb{H})_{\mathrm{sa}}$, $S_9 \cong M_2(\mathbb{O})_{\mathrm{sa}}$

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- But $M_3(\mathbb{O})_{\mathrm{sa}}$ is *not* special. It is *exceptional*.

Are there other exceptional Jordan algebras?

Classification of Jordan algebras

Jordan-von Neumann-Wigner, 1934

Let E be a real finite-dimensional formally real Jordan algebra. Then $E \cong E_1 \oplus \cdots \oplus E_k$ where each E_j is equal to one of the following non-isomorphic formally real Jordan algebras:

- The set of real numbers $\mathbb{R}$.
- The algebras $M_n(\mathbb{F})_{\mathrm{sa}}$ where $F = \mathbb{R}$, $F = \mathbb{C}$ or $F = \mathbb{H}$ for $n \geqslant 3$.
- The spin factors S_n for $n \geqslant 2$.
- The exceptional Jordan algebra $M_3(\mathbb{O})_{\mathrm{sa}}$.

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- The exceptional Jordan algebra $M_3(\mathbb{O})_{\mathrm{sa}}$.

Corollary

E is isomorphic to the direct sum of a special Jordan algebra and some number of copies of $M_3(\mathbb{O})_{\mathrm{sa}}$.

Quantum logic

Instead of looking at $B(\mathscr{H})$ or $B(\mathscr{H})_{\mathsf{sa}}$, consider the sharp measurements $P(\mathscr{H})$.

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- For each $p \in P(\mathscr{H})$ we have a *complement* $p^\perp := 1 - p$.
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- This satisfies $p \vee p^\perp = 1, p \wedge p^\perp = 0$ and $(p^\perp)^\perp = p$.
- Can define *orthogonality* in terms of complement: $p \perp q \iff q \leqslant p^\perp$.
- Additionally we have: $p \perp q \implies p = q^\perp \wedge (p \vee q)$.
- $P(\mathscr{H})$ is an *orthomodular* lattice (OML).

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- Every $p \in P(\mathscr{H})$ has a 'dimension': a minimal number of atoms $q_1, \ldots, q_n$ such that $p = q_1 \vee \cdots \vee q_n$.

Covering law

Definition

Let $a, b \in L$ be elements of a lattice. We say b *covers* a when $a \leqslant b$ and if $a \leqslant c \leqslant b$ then $a = c$ or $c = b$.

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$P(\mathscr{H})$ is a complete irreducible atomistic OML with the covering property. Does this characterise it?
Sort of...

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- A *Hermitian form* over a $\mathbb{K}$-vector space V is a left-$\mathbb{K}$-linear form $(v, w) \mapsto \langle v, w \rangle$ such that $\langle v, w \rangle^* = \langle w, v \rangle$.
- It is *non-degenerate* when $\langle w, v \rangle = 0$ for all w implies $v = 0$.
- For $S \subseteq V$, we denote its *orthocomplement* by $S^\perp$. Call S *closed* when $S^{\perp\perp} = S$.
- Write $P(V) := \{S \subseteq V \text{ closed}\}$.

Generalised Hilbert spaces

Proposition

Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ and let V be a $\mathbb{K}$-vector space with a non-degenerate Hermitian form. Then the following are equivalent:

- V is a Hilbert space (i.e. is complete in the norm).
- $P(V)$ is orthomodular.
- $S \oplus S^{\perp} = V$ for all $S \in P(V)$.

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Definition

A *generalised Hilbert space* is a $\mathbb{K}$-vector space with a
non-degenerate Hermitian form
such that $S \oplus S^{\perp} = V$ for all $S \in P(V)$.

Piron and Sòler

Piron's theorem, 1964

Let L be a complete irreducible atomistic OML with the covering property, that contains a set of at least 4 orthogonal atoms. Then there is a generalised Hilbert space V such that $L \cong P(V)$.

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Corollary

Let L be a complete irreducible atomistic OML of infinite rank and satisfying the covering property. Then $L \cong P(\mathscr{H})$ where $\mathscr{H}$ is a real, complex or quaternionic Hilbert space.

Conclusion

We can get operators on a (complex) Hilbert space via:

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We then get the correct states via Gleason, and the correct dynamics via Wigner+Stone.
It remains to find compelling principles to derive 1-3.