Quantum Compilation using the ZX-calculus

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Used in:

- Quantum circuit optimisation and compilation
- Measurement-based quantum computation
- Surface codes and lattice surgery
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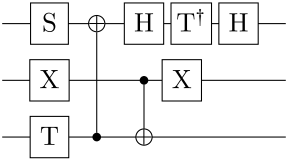
It is also a convenient tool for day-to-day quantum reasoning

Further Reading

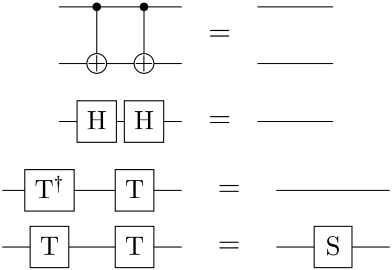
For references and details see:
ZX-calculus for the working quantum computer scientist
`https://arxiv.org/abs/2012.13966`

For a book-length introduction see:
Picturing Quantum Processes: A First Course in Quantum Theory and Diagrammatic Reasoning
by Bob Coecke and Aleks Kissinger

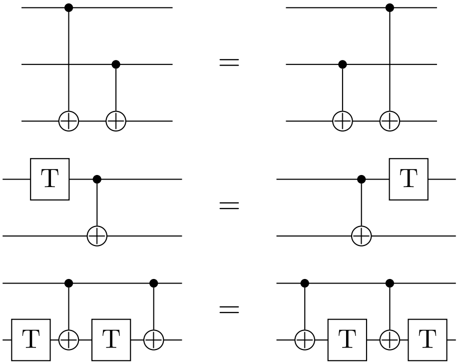
Quantum circuits



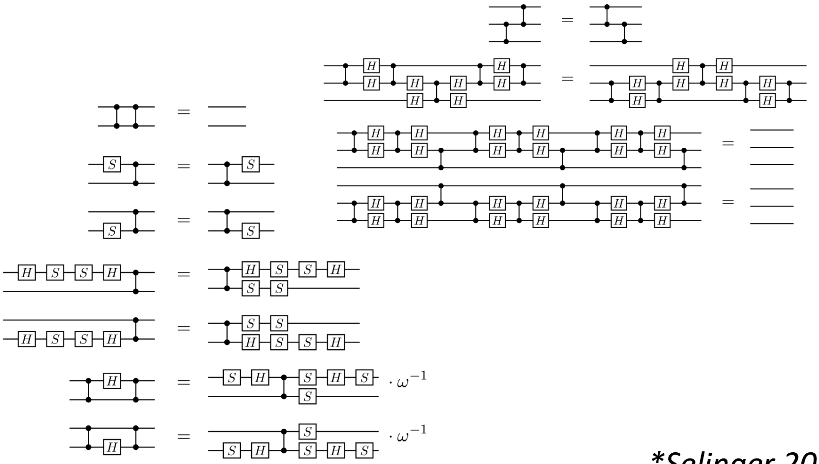
Circuit identities



Gate commutation

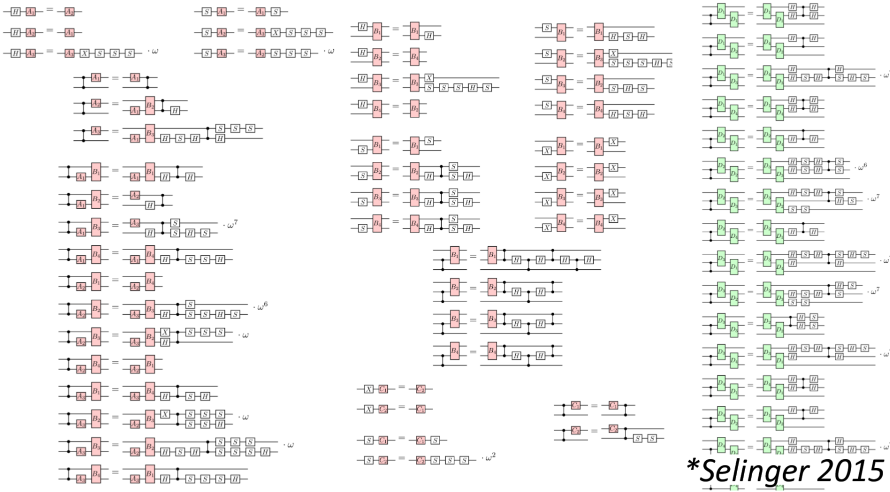


More circuit equalities



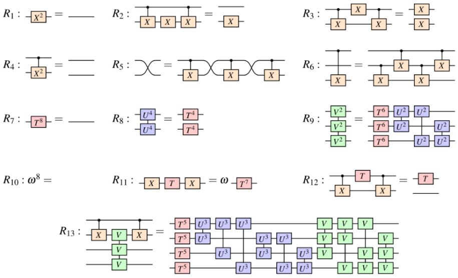
**Selinger 2015*

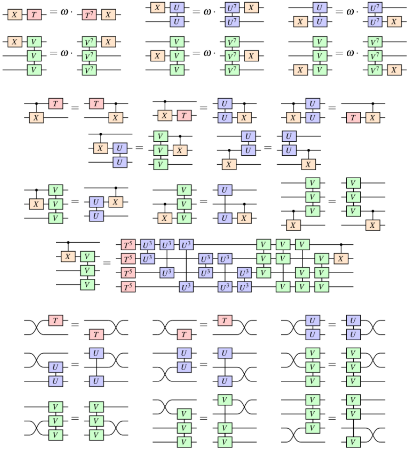
And more circuit equalities



*Selinger 2015

And even more circuit equalities





*Amy, Chen, & Ross 2018

Quantum circuits bad!

Why is this so terrible?

- Choice of gates is a bit arbitrary.
- The notation is not “quantum native”.
- Wires are rigid going from left-to-right.

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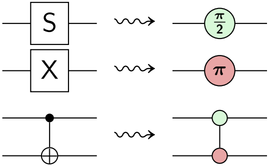
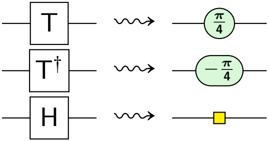
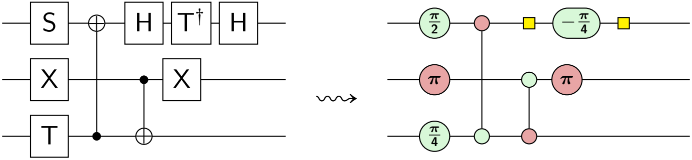
The ZX-calculus essentially gets rid of these problems.

ZX-diagrams

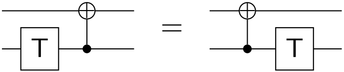
On a surface level, ZX-diagrams are alternative notation to circuits



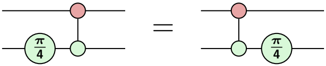




Circuit identity in ZX



becomes



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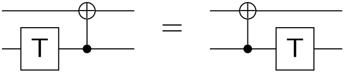
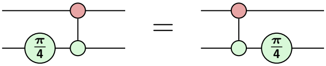
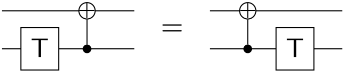


becomes

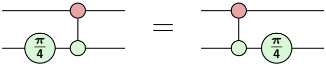


dots of same colour commute through each other

Circuit identity in ZX



becomes



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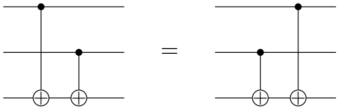

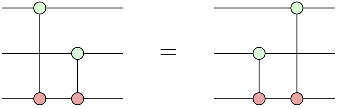

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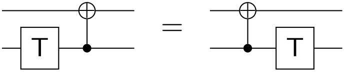

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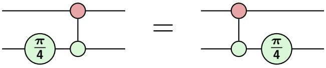

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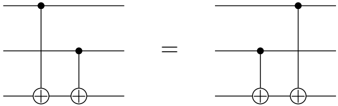
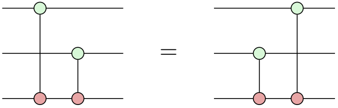
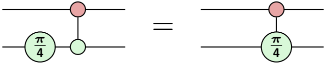




More fundamental rule: *dots of same colour fuse*



States in ZX

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad |+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad |-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

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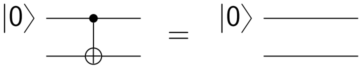

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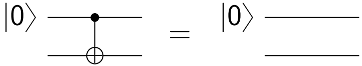

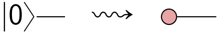

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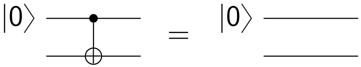
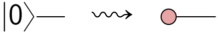
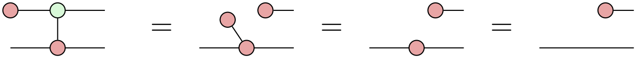




single-wire dot copies through opposite-coloured dot



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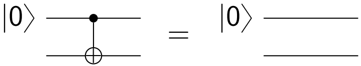

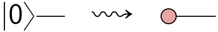

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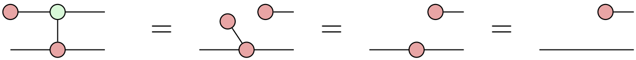
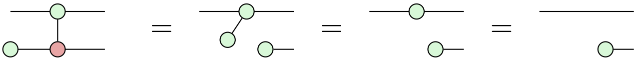


all rules hold with colours interchanged



Now let's formally introduce ZX-diagrams

Spiders

What gates are to circuits, *spiders* are to ZX-diagrams.

Spiders

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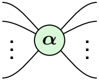
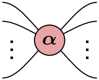
Z-spider

$$|0\cdots0\rangle\langle0\cdots0| + e^{i\alpha}\,|1\cdots1\rangle\langle1\cdots1|$$



X-spider

$$|+\cdots+\rangle\langle+\cdots+| + e^{i\alpha}\,|-\cdots-\rangle\langle-\cdots-|$$



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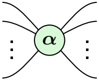

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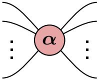
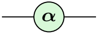
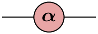


For example:



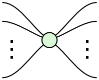
$$= |0\rangle\langle 0| + e^{i\alpha}\,|1\rangle\langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & e^{i\alpha} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$



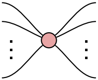
$$= |+\rangle\langle +| + e^{i\alpha}\,|-\rangle\langle -| = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{1}{2}e^{i\alpha}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Spiders cont.

If $\alpha = 0$ we drop the label:



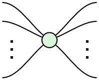
$$= \quad |0\cdots 0\rangle\langle 0\cdots 0| + |1\cdots 1\rangle\langle 1\cdots 1|$$

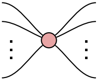


$$= \quad |+\cdots +\rangle\langle +\cdots +| + |-\cdots -\rangle\langle -\cdots -|$$

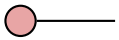
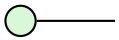
Spiders cont.

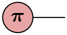
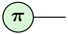
If $\alpha = 0$ we drop the label:

 $= \quad |0\cdots0\rangle\langle0\cdots0| + |1\cdots1\rangle\langle1\cdots1|$

 $= \quad |+\cdots+\rangle\langle+\cdots+| + |-\cdots-\rangle\langle-\cdots-|$

Example:

 $= \quad |+\rangle + |-\rangle \; = \sqrt{2}\,|0\rangle$  $= \quad |0\rangle + |1\rangle \; = \sqrt{2}\,|+\rangle$

 $= \quad |+\rangle - |-\rangle \; = \sqrt{2}\,|1\rangle$  $= \quad |0\rangle - |1\rangle \; = \sqrt{2}\,|-\rangle$

We ignore these non-zero scalar factors

Formal composition

Spiders can be composed in two ways.

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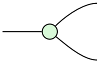
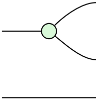
Horizontal composition gives tensor product:



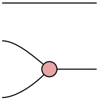
$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$



$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Formal composition

Other tensor product:

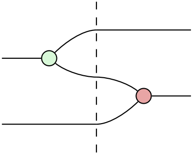


$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Formal composition

Horizontal composition is regular composition of linear maps:

 $=$

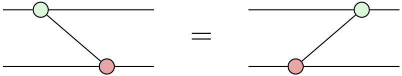
$$\frac{1}{\sqrt{2}}\begin{pmatrix}1&0&0&1&0&0&0&0\\0&1&1&0&0&0&0&0\\0&0&0&0&1&0&0&1\\0&0&0&0&0&1&1&0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$$

Building ZX-diagrams

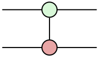
Any ZX-diagram is built by simply iterating these vertical and horizontal compositions

Symmetries

Note:



Hence, we may write



Symmetries

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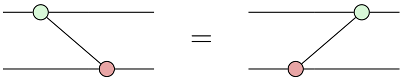
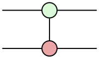
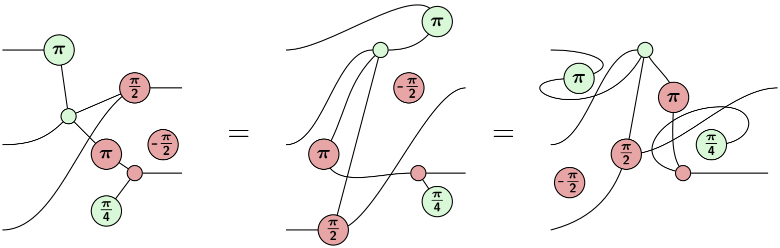


Hence, we may write



In general: *only connectivity matters*



ZX-diagrams summary

- Two types of generators: Z-spiders and X-spiders
- Can compose both horizontally and vertically
- Wires can connect every which way

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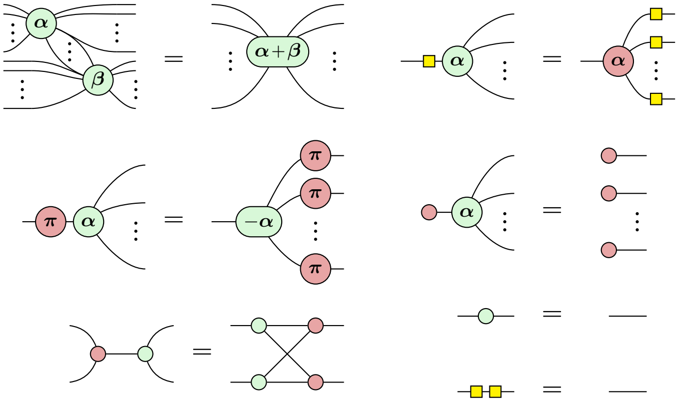
How powerful are ZX-diagrams as a representation?

Theorem

ZX-diagrams are *universal*: any linear map between qubits can be represented as a ZX-diagram.

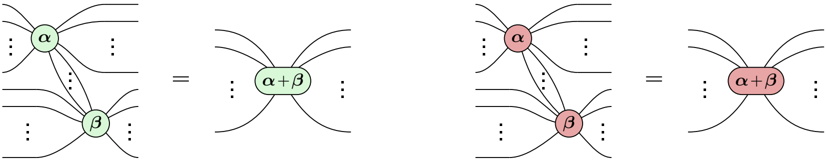
So far it's just notation. What can we do with it?

Rules for ZX-diagrams: The ZX-calculus



$\alpha, \beta \in [0, 2\pi]$

Spider fusion



Connected spiders of same colour fuse

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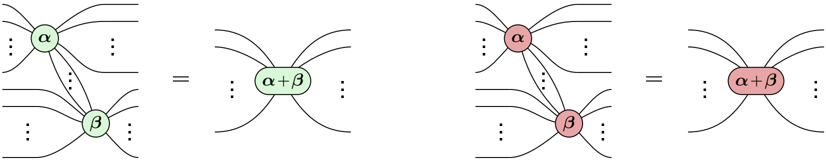

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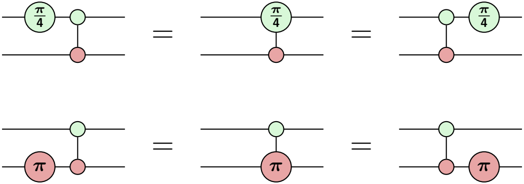
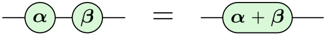
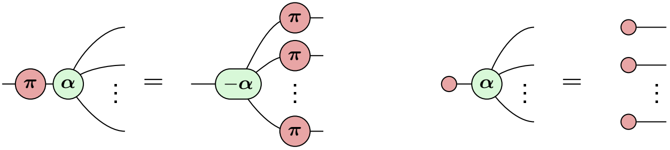




State and pi-copy



π's and states copy through the other colour

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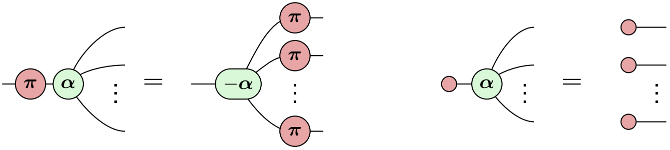

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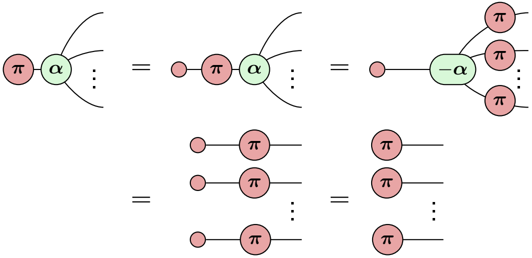
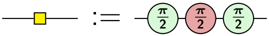
Combining rules:



Hadamards and colour-changing

Definition of Hadamard in ZX:



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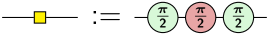
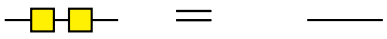
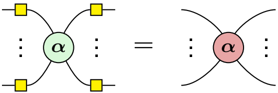


Rules:





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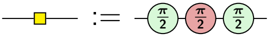

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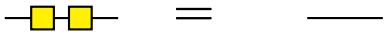
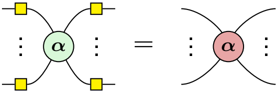
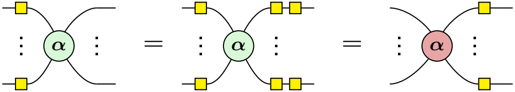




Derived rule: *commuting Hadamards changes colour*



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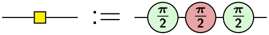

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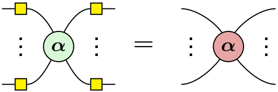

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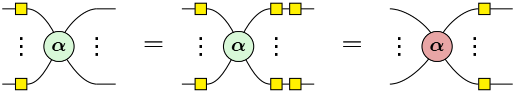
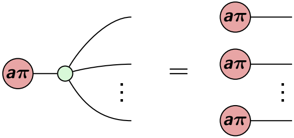
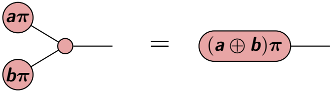


Consequence: *Everything in ZX holds with colours reversed*

Bialgebra

Z-spiders act like COPY; X-spiders act like XOR:





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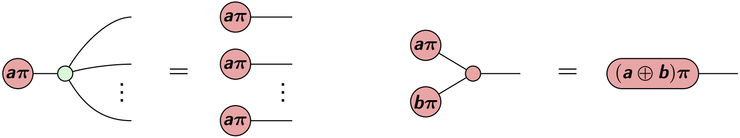

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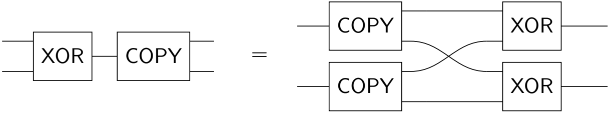

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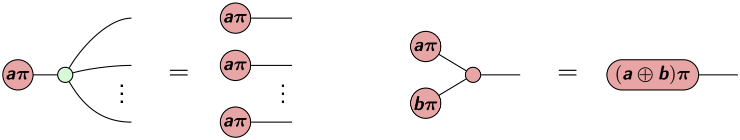

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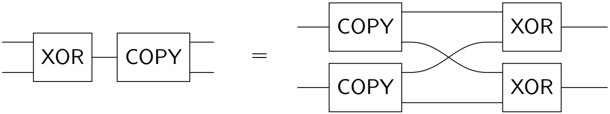
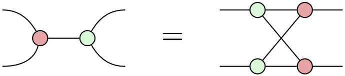
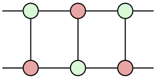


Hence:



Three CNOTs make SWAP



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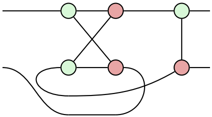

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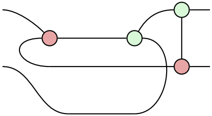

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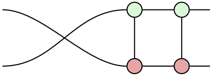

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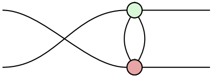

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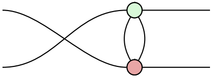

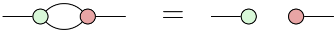

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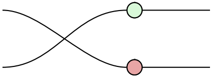

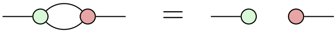

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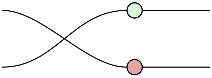

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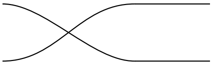

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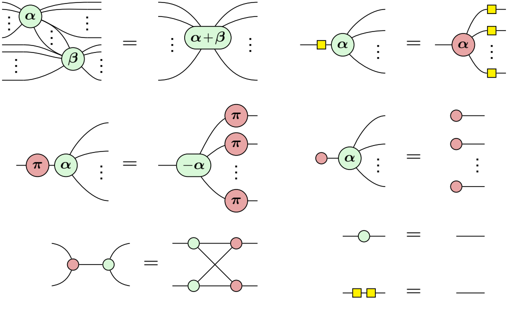
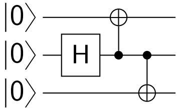


$\alpha, \beta \in [0, 2\pi]$

- All derivations hold in any orientation
- All derivations hold with colours interchanged
- All derivations hold with phases negated

Example 1: GHZ-preparation circuit

Recall that the GHZ-state is $|000\rangle + |111\rangle$.
The following circuit creates a GHZ-state:



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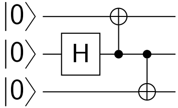
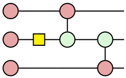


Proof:



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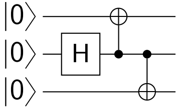

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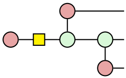

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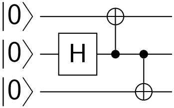

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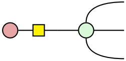

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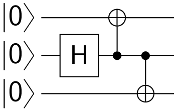

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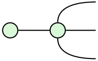

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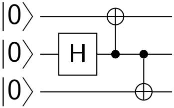

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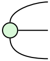
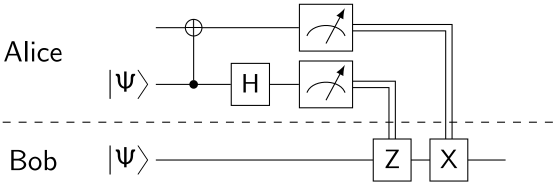


Example 2: Teleportation

Let $|\Psi\rangle$ represent a side of a Bell state.
Then this is the standard quantum teleportation protocol:



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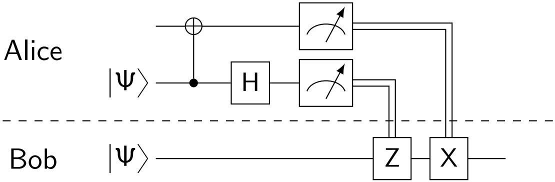

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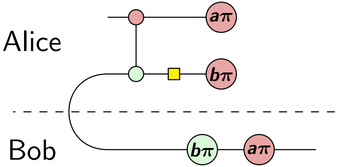

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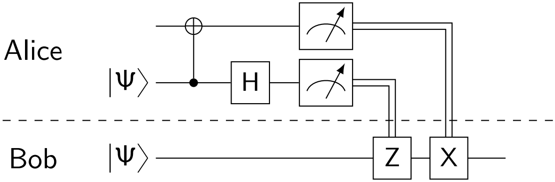

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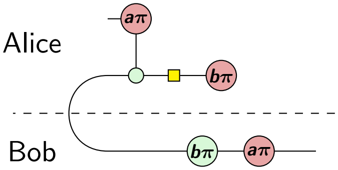

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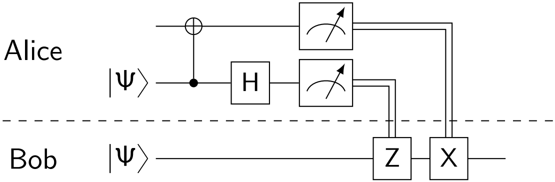

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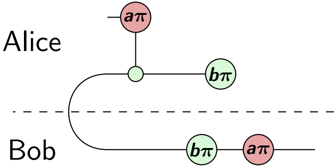

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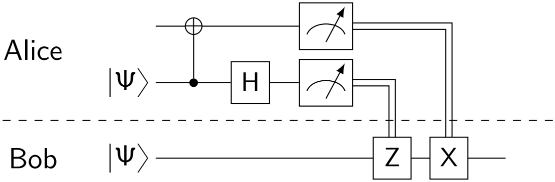

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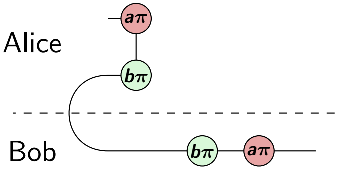

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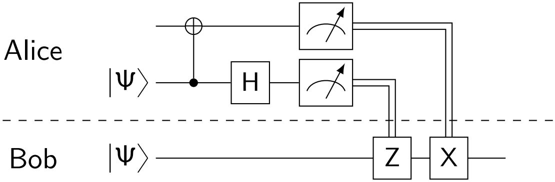

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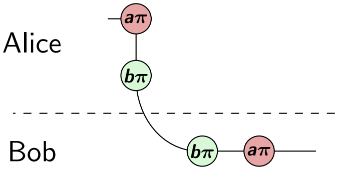

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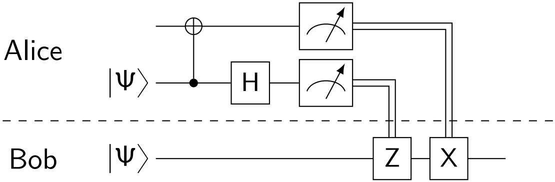

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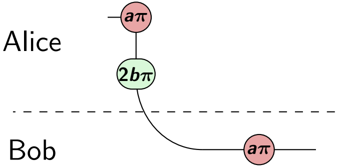

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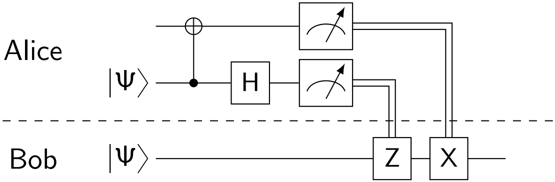

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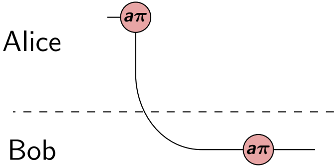

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Let $|\Psi\rangle$ represent a side of a Bell state.
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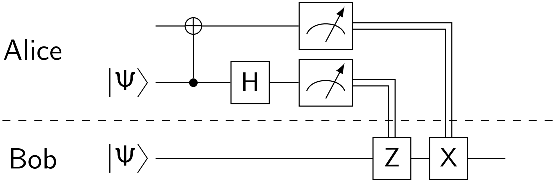

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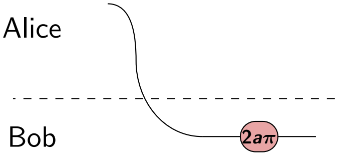

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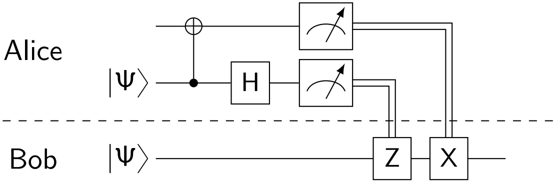

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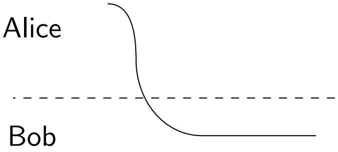

Now let's look at specific use-cases of ZX

Use-case of ZX #1: Clifford computation

Gottesman-Knill theorem

A quantum circuit of Cliffords can be efficiently classically simulated.

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Can we prove this using ZX?

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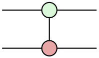
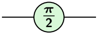
- As a ZX-diagram, a Clifford map only has phases multiple of $\frac{\pi}{2}$.

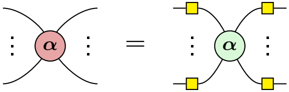
CNOT =  S = 

H = 

- Conversely, ZX-diagrams with phases multiple of $\frac{\pi}{2}$ are Clifford.

Graph-like diagrams

Every ZX-diagram can be reduced to a *graph-like* diagram:

- Only Z-spiders and Hadamards: 

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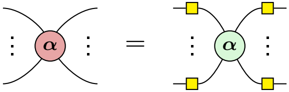
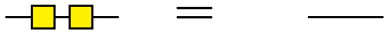
- Only Z-spiders and Hadamards: 

- Cancel adjacent hadamards: 

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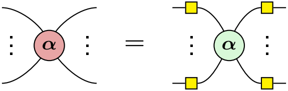
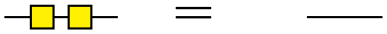
- Only Z-spiders and Hadamards: 

- Cancel adjacent hadamards: 

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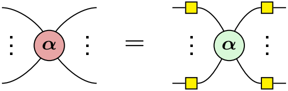
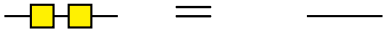
- Only Z-spiders and Hadamards: 

- Cancel adjacent hadamards: 

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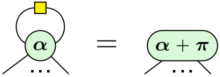
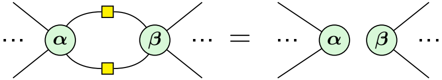
- No self-loops or multiple edges:





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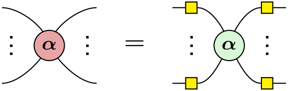
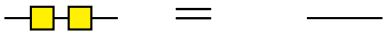
- Only Z-spiders and Hadamards: 
- Cancel adjacent hadamards: 
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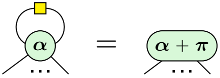
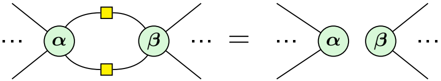
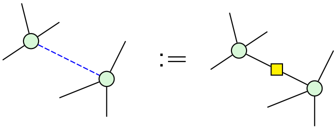




- View all Hadamards as a type of *edge*:

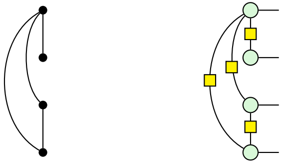


Graph states

A graph-like diagram is a *graph state* when

- it has no inputs,
- every spider is connected to a unique output,
- all phases are zero.

Example:



G $\qquad$ $|G\rangle$

Graph-theoretic rewriting

We've transformed ZX-diagrams into simple undirected graphs so we can view rewrites graph-theoretically.

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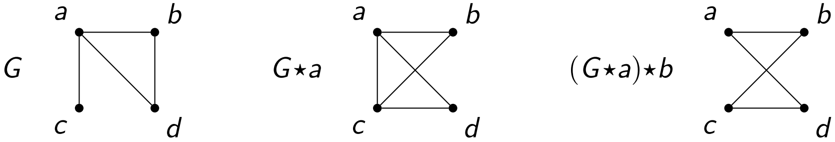
Local complementation



$G \star a := G$, but with connectivity of neighbours of a complemented.

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Local complementation

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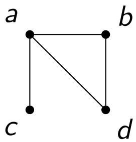
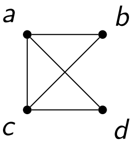


$G\star a$



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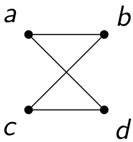
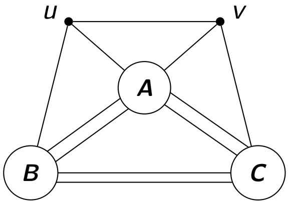


$G \star a := G$, but with connectivity of neighbours of a complemented.

A *pivot* on edge uv is $G \wedge uv := G \star u \star v \star u$.

G



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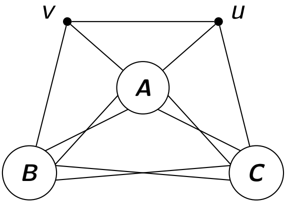
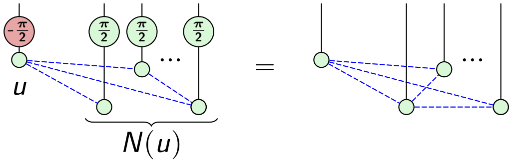


Local complementation on graph states

An lcomp on graph state can be implemented using *local Cliffords*:



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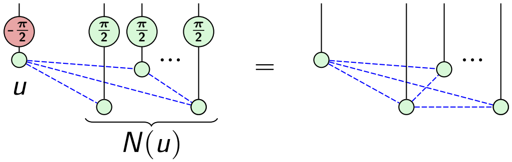
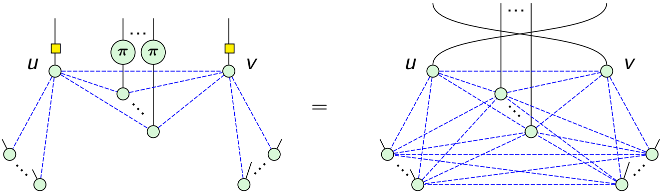
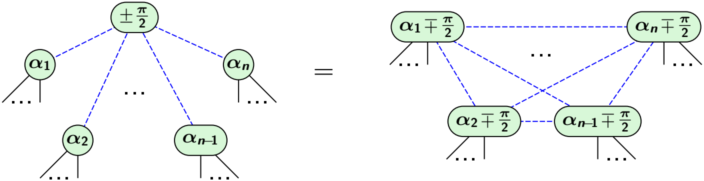


Same goes for pivot:

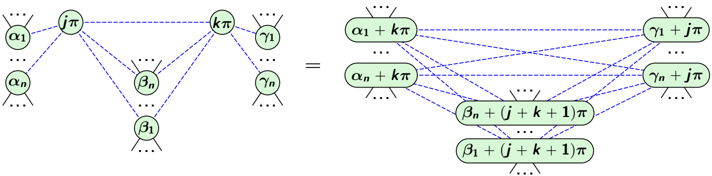


Removing vertices

Remove vertex by lcomp:



Similarly, using pivot:



Clifford simplification

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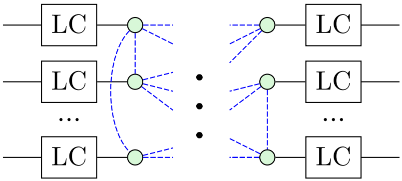
Other consequence: Clifford circuit reduced to



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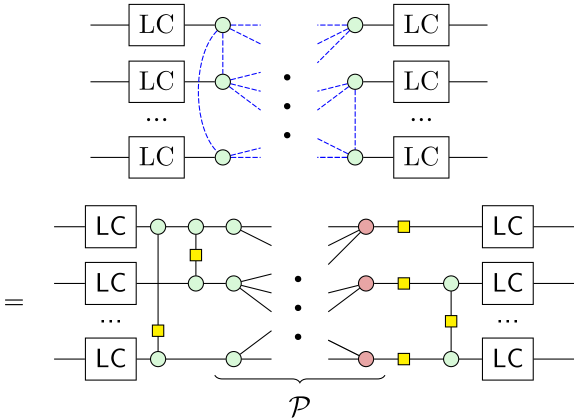
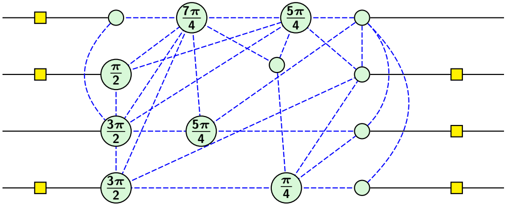


Normal form of layers:

H+S+CZ+CNOT+H+CZ+S+H

Simplifying general circuits

Example result after simplification:



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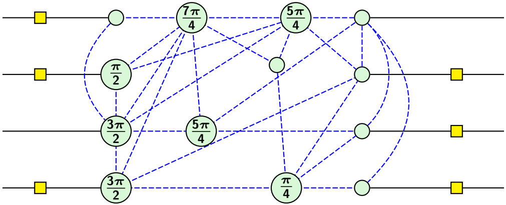

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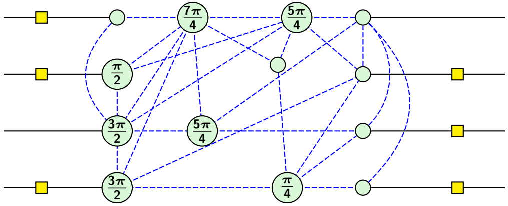

Problem: does not look a circuit.
Solution: all rewrites preserve *gflow*.

- Duncan, Perdrix, Kissinger, vdW (2019). *Graph-theoretic Simplification of Quantum Circuits with the ZX-calculus*.
- Backens, Miller-Bakewell, de Felice, Lobski, vdW (2020). *There and back again: A circuit extraction tale*.

Non-Clifford optimisation

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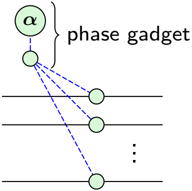
Additional rules for *phase gadgets*:



$$:: \quad \left|x_1, \ldots, x_n\right\rangle \mapsto e^{i\alpha(x_1 \oplus \ldots \oplus x_n)} \left|x_1, \ldots, x_n\right\rangle$$

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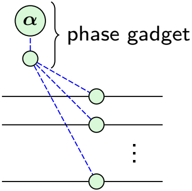

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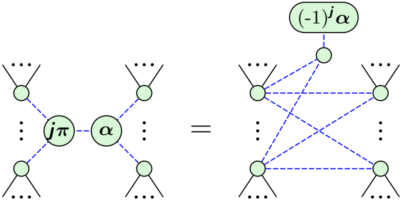

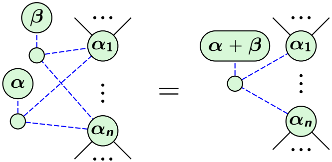

Kissinger, vdW 2019: *Reducing T-count with the ZX-calculus*

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- Combining with TODD [Heyfron & Campbell 2018] we improved T-counts for 20/36 circuits.
- Note: [Zhang & Chen 2019] use a different method that achieves nearly identical T-counts.

Circuit equality verification

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- Compose optimised circuit with adjoint of original circuit
- Simplify
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Using this method found mistake in other peer-reviewed optimiser.

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- Treat ZX-diagram as tensor network and contract.
- Use stabiliser decomposition of magic states to write as sum of simpler diagrams.

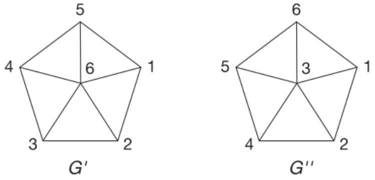


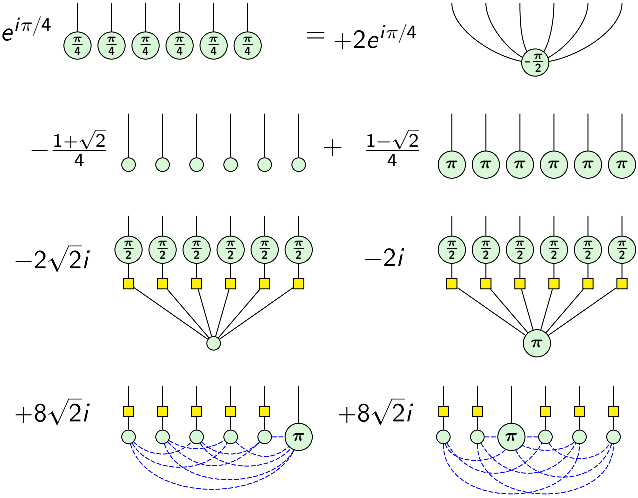
FIG. 3. Graphs G' and G'' used in the definition of stabilizer states ϕ' and ϕ''; see Eq. (11).

$$
\begin{aligned}
|H^{\otimes 6}\rangle = (-16 + 12\sqrt{2})|B_{6,0}\rangle + (96 - 68\sqrt{2})|B_{6,6}\rangle \\
+ (10 - 7\sqrt{2})|E_6\rangle + (-14 + 10\sqrt{2})|O_6\rangle \\
+ (7 - 5\sqrt{2})Z^{\otimes 6}|K_6\rangle + (10 - 7\sqrt{2})|\phi'\rangle \\
+ (10 - 7\sqrt{2})|\phi''\rangle,
\end{aligned} \tag{11}
$$

where

$$
|\phi'\rangle = \prod_{(i,j)\in E'} \Lambda(Z)_{i,j}|O_6\rangle \quad \text{and} \quad |\phi''\rangle = \prod_{(i,j)\in E''} \Lambda(Z)_{i,j}|O_6\rangle.
$$

Source: Sergey Bravyi, Graeme Smith, and John A Smolin. *Trading classical and quantum computational resources* (2016).


$e^{i\pi/4}$
$\frac{\pi}{4}$ $\frac{\pi}{4}$ $\frac{\pi}{4}$ $\frac{\pi}{4}$ $\frac{\pi}{4}$ $\frac{\pi}{4}$
$= +2e^{i\pi/4}$
$-\frac{\pi}{2}$
$-\frac{1+\sqrt{2}}{4}$
$+$
$\frac{1-\sqrt{2}}{4}$
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π
$+8\sqrt{2}i$
π
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π

Circuit simulation with ZX-calculus

1. Write circuit+state as ZX-diagram.
2. Simplify using ZX-calculus rules.
3. Replace magic states by stabilizer decomposition.
4. Repeat.
5. ...
6. Profit!

Early results looks like this could give major benefit

Stuff I didn't talk about

- CNOT optimisation
- Relationship to MBQC and lattice surgery
- Circuit routing
- Applications in tensor networks

Conclusion

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- It allows you to graphically do many things

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Thank you for your attention

vdW 2020, arXiv:2012.13966.
ZX-calculus for the working quantum computer scientist