Sequential Measurement characterises Quantum Theory

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In quantum theory $a \,\&\, b := \sqrt{a} b \sqrt{a}$ and $a \mid b$ when $ab = ba$.

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- But it can be recovered with one additional assumption.

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V's with $\omega(c) \leq 1 \; \forall\omega \implies c \leq 1$ are *order unit spaces*.

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Resistance to noise: $a \mapsto a \,\&\, b$ is continuous.

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- Not enough to characterise $b \mapsto \sqrt{a} b \sqrt{a}$.

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- ⇒ Positive cone of V is homogeneous.

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Sequential effect spaces are just Jordan algebras in disguise!

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Sequential Measurement Characterises Quantum Theory

A locally tomographic generalised probabilistic theory of sequential effect spaces embeds into $\mathbf{CStar}_{\mathrm{CP}}$.

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The full theorem

A finite-dimensional order unit space V with $E = \{a \in V \ ; \ 0 \leq a \leq 1\}$ and $\& : E \times E \to E$ such that

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- $1 \,\&\, a = a$ and $a \,\&\, b = 0 \implies b \,\&\, a = 0$
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- $a \,|\, b \implies a \,|\, b^\perp$ and $a \,|\, b, c \implies a \,|\, (b + c)$, $a \,|\, (b \,\&\, c)$
- it has locally tomographic composite $V \otimes V$
- and $(a_1 \otimes a_2) \,\&\, (b_1 \otimes b_2) = (a_1 \,\&\, b_1) \otimes (a_2 \,\&\, b_2)$,

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- $a \,|\, b \implies a \,\&\, (b \,\&\, c) = (a \,\&\, b) \,\&\, c$
- $a \,|\, b \implies a \,|\, b^\perp$ and $a \,|\, b, c \implies a \,|\, (b + c),\; a \,|\, (b \,\&\, c)$
- it has locally tomographic composite $V \otimes V$
- and $(a_1 \otimes a_2) \,\&\, (b_1 \otimes b_2) = (a_1 \,\&\, b_1) \otimes (a_2 \,\&\, b_2)$,

then V is a C*-algebra.

Minimality of conditions

The full theorem

A finite-dimensional order unit space V with
$E = \{a \in V\ ;\ 0 \leq a \leq 1\}$ and $\& : E \times E \to E$ such that

- $a \,\&\, (b + c) = a \,\&\, b + a \,\&\, c$ and $a \mapsto a \,\&\, b$ is continuous
- $1 \,\&\, a = a$ and $a \,\&\, b = 0 \implies b \,\&\, a = 0$
- $a \,|\, b \implies a \,\&\, (b \,\&\, c) = (a \,\&\, b) \,\&\, c$
- $a \,|\, b \implies a \,|\, b^{\perp}$ and $a \,|\, b, c \implies a \,|\, (b + c),\ a \,|\, (b \,\&\, c)$
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then V is a C*-algebra.

Minimality of conditions

The full theorem

A finite-dimensional order unit space V with $E = \{a \in V \; ; \; 0 \leq a \leq 1\}$ and $\& : E \times E \to E$ such that

- $a \,\&\, (b + c) = a \,\&\, b + a \,\&\, c$ and $a \mapsto a \,\&\, b$ is continuous
- $1 \,\&\, a = a$ and $a \,\&\, b = 0 \implies b \,\&\, a = 0$
- $a \mid b \implies a \,\&\, (b \,\&\, c) = (a \,\&\, b) \,\&\, c$
- $a \mid b \implies a \mid b^\perp$ and $a \mid b, c \implies a \mid (b + c)$, $a \mid (b \,\&\, c)$
- it has locally tomographic composite $V \otimes V$
- and $(a_1 \otimes a_2) \,\&\, (b_1 \otimes b_2) = (a_1 \,\&\, b_1) \otimes (a_2 \,\&\, b_2)$,

then V is a C*-algebra.

Minimality of conditions

The full theorem

A finite-dimensional order unit space V with
$E = \{a \in V\ ;\ 0 \leq a \leq 1\}$ and $\&: E \times E \to E$ such that

- $a \,\&\, (b + c) = a \,\&\, b + a \,\&\, c$ and $a \mapsto a \,\&\, b$ is continuous
- $1 \,\&\, a = a$ and $a \,\&\, b = 0 \implies b \,\&\, a = 0$
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- and $(a_1 \otimes a_2) \,\&\, (b_1 \otimes b_2) = (a_1 \,\&\, b_1) \otimes (a_2 \,\&\, b_2)$,

then V is a C*-algebra.

Conclusion

Finite-dimensional order unit space
Continuous sequential product

+

Euclidean Jordan algebra

Conclusion

Finite-dimensional order unit space
Continuous sequential product

——————————————— +

Euclidean Jordan algebra
Local Tomography

——————————————— +

C*-algebra

Conclusion

Finite-dimensional order unit space
Continuous sequential product

———————————— +

Euclidean Jordan algebra
Local Tomography

———————————— +

C*-algebra

$\Rightarrow$ **Sequential Measurement characterises Quantum Theory**

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Gudder & Greechie (2002): *Sequential Products on Effect Algebras*

vdW (2018): *Three Characterisations of the Sequential Product*
arXiv:1803.08453

vdW (2018): *Sequential Measurement Characterises Quantum Theory*
arXiv:1803.11139

Thank you for your attention