The algebraic structure of quantum effects

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In this talk

- Hilbert spaces and quantum logic.
- From orthomodular lattices to effect algebras.
- Some cool things you can do with effect algebras.

Hilbert spaces and quantum logic

- Quantum system is modelled by complex Hilbert space $\mathcal{H}$.
- Propositions modelled by closed subspaces.
- Equivalently: positive idempotent operators $P : \mathcal{H} \to \mathcal{H}$.
- What is the structure of the set of projections?

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- Orthomodularity: $P \leqslant Q \implies P \vee (P^\perp \wedge Q) = Q$.

This is the ‘classical’ description of quantum logic. But what if we want to allow fuzziness?

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- *partial* commutative associative $\owedge$,
- with $a \owedge 0 = a$,
- and $\forall a$ unique $a^\perp$ with $a \owedge a^\perp = 1$,
- such that $a \perp 1$ implies $a = 0$.

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Examples

- $[0, 1]$ with $a^\perp := 1 - a$.
- An orthomodular lattice: addition defined when $a \wedge b = 0$ and then $a \ovee b := a \vee b$.
- $\mathbf{Cstar}(\mathbb{C}, \mathfrak{A}) \cong [0, 1]_{\mathfrak{A}}$ with $a^\perp := 1 - a$.

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- Hence, there is a resulting *Kalmbach monad* K : **BPos** $\rightarrow$ **BPos**.
- The Eilenberg-Moore algebras of this monad are precisely effect algebras.

Usage 1: Generalising GPTs

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- In last talk John Selby talked about *generalised probabilistic theories* (GPTs) as generalisation of quantum theory.
- But we can also generalise GPTs
- Instead of convex sets, we get effect algebras.
- Instead of a GPT we get an *effectus*.
- See Kenta Cho's thesis
  Effectuses in Categorical Quantum Foundations.

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Examples

- $[0, 1]$.
- Any Boolean algebra: $a \ovee b := a \vee b$, $a \cdot b := a \wedge b$.
- $\{f : X \to [0, 1] \text{ continuous}\}$ for a compact Hausdorff space X (i.e. unit interval of commutative unital C*-algebra).

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Equivalent definition

In ωEA increasing sequences $a_1 \leqslant a_2 \leqslant \dots$ have supremum.

ω-effect-monoids

Theorem (Westerbaan, Westerbaan & vdW, LICS'20)

An ω-effect-monoid M embeds into $M_1 \oplus M_2$ where

- M_1 is a ω-complete Boolean algebra
- $M_2 = \{f : X \to [0,1] \text{ cont.}\}$ for basically disconnected X.

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ω-effect-monoids are commutative.

Call M *irreducible* when $M \cong M_1 \oplus M_2$ implies $M_i = \{0\}$.

Corollary

The only irreducible ω-effect-monoids are $\{0\}$, $\{0,1\}$ and $[0,1]$.

Usage 3: rederiving quantum theory

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Definition (Gudder & Greechie, 2002)

Let E be effect algebra with operation $\& : E \times E \to E$.
Write $a \,|\, b$ when $a \,\&\, b = b \,\&\, a$.
$(E, \&)$ is a *sequential effect algebra* when:

- $a \,\&\, (b + c) = a \,\&\, b + a \,\&\, c$
- $1 \,\&\, a = a$ and if $a \,\&\, b = 0$ then also $b \,\&\, a = 0$.
- If $a \,|\, b$ then $a \,\&\, (b \,\&\, c) = (a \,\&\, b) \,\&\, c$.
- If $a \,|\, b$ then $a \,|\, b^{\perp}$, and if also $a \,|\, c$ then $a \,|\, (b + c)$ & $a \,|\, (b \,\&\, c)$.

From sequential effect algebra to quantum theory

Theorem (vdW, 2019)

Let V be finite-dimensional order unit space,
such that $E := [0,1]_V$ has norm-continuous sequential product.
Then V is a Euclidean Jordan algebra.

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Theorem (Westerbaan, Westerbaan, vdW, 2020)

Let E be directed-complete effect algebra with *normal* sequential product. Then $E \cong E_1 \oplus E_2 \oplus E_3$ where

- E_1 is complete Boolean algebra,
- $E_2 := [0,1]_V$ for V an order unit space,
- and E_3 an *almost-convex* effect algebra.

Conclusion

- Effect algebras generalise quantum logic to allow for fuzziness.
- Can be used to talk abstractly about probabilities.
- Can be used to rederive quantum mechanics.

Thank you for your attention!

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The three types of normal sequential effect algebras