A Categorical Construction of the Real Unit Interval

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Yes! Using ω-effect-monoids

Main result in brief

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Category of ω-effect-monoids is monadic over category of bounded posets.

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Theorem (Westerbaan[2] & vdW, 2020)

The only irreducible ω-effect-monoids are $\{0\}$, $\{0,1\}$ and $[0,1]$.

So: $[0,1]$ is unique non-initial, non-final irreducible Eilenberg-Moore algebra of particular monad over bounded posets.

Posets and Kalmbach extension

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Let P be a poset and $a, b \in P$.

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What are the algebras of the resulting *Kalmbach monad*?

Effect algebras

Definition

An **effect algebra** $(E, \ovee, 0, 1)$ has

- *partial* commutative associative $\ovee$,
- with $a \ovee 0 = a$,
- and $\forall a$ unique $a^{\perp}$ with $a \ovee a^{\perp} = 1$,
- such that $a \perp 1$ implies $a = 0$.

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Examples

- $[0, 1]$ with $a^\perp := 1 - a$.
- A Boolean algebra: addition defined when $a \wedge b = 0$ and then $a \ovee b = a \vee b$. $a^\perp$ is regular negation.
- $\mathbf{Cstar}(\mathbb{C}, \mathfrak{A}) \cong [0, 1]_{\mathfrak{A}}$ with $a^\perp := 1 - a$.

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Theorem

$\mathbf{EA} \cong \mathbf{BPos}^K$.

Effect monoids

So $[0,1]$ is an effect algebra. But it also has a multiplication.
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An **effect monoid** $(M, \ovee, 0, 1, \cdot)$ is effect algebra with associative distributive multiplication:

$$(a \ovee b) \cdot c = (a \cdot c) \ovee (b \cdot c) \qquad c \cdot (a \ovee b) = (c \cdot a) \ovee (c \cdot b)$$

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Examples:

- $[0,1]$.
- Any Boolean algebra: $a \ovee b := a \vee b$, $a \cdot b := a \wedge b$.
- $\{f : X \to [0,1] \text{ continuous}\}$ for a compact Hausdorff space X (i.e. unit interval of commutative unital C*-algebra).

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Equivalent definition

In ωEA increasing sequences $a_1 \leqslant a_2 \leqslant \ldots$ have supremum.

ω-effect-monoids

Theorem (Westerbaan, Westerbaan & vdW, LICS'20)

An ω-effect-monoid M embeds into $M_1 \oplus M_2$ where

- M_1 is an ω-complete Boolean algebra
- $M_2 = \{f : X \to [0,1] \text{ cont.}\}$ for basically disconnected X.

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ω-effect-monoids are commutative.

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Corollary

ω-effect-monoids are commutative.

Call M *irreducible* when $M \cong M_1 \oplus M_2$ implies $M_i = \{0\}$.

Corollary

The only irreducible ω-effect-monoids are $\{0\}$, $\{0,1\}$ and $[0,1]$.

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So $[0, 1]$ is an irreducible monoid in an EM-category over **BPos**.

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ω**EM** is also also monadic over **BPos**.

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There is a monad T over **BPos** such that $[0, 1]$ is the unique irreducible non-initial, non-final T-algebra.

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There is a monad T over **BPos** such that $[0, 1]$ is the unique irreducible non-initial, non-final T-algebra.
Furthermore, $\mathbf{BPos}^T \cong \omega\mathbf{EM}$ and these algebras have

- a partial order,
- a (partially defined) countable addition,
- a negation,
- and a multiplication.

So we have captured what is special about $[0, 1]$.

Conclusion and open questions

- We've found a categorical construction of $[0, 1]$.
- This captures its relevant structure as a space of probabilities.
- Observation: $\{0\}$ is final and $\{0, 1\}$ is initial,
  while $[0, 1]$ is *just right*.
- Result proven using Beck's monadicity theorem. Can we do it constructively?

Thank you for your attention!

vdW 2021, arXiv: 2106.10094
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