Quantum Theory from First Principles
Lecture 4

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Previously

- Reconstructing properties of quantum theory with ‘partial reconstructions’.

Today

- A reconstruction of my own, based on well-behavedness of sequential measurement.
- Reconstructing quantum theory without even assuming the basic probabilistic calculus, i.e. that probabilities are numbers in $[0, 1]$?

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In quantum theory $a \,\&\, b := \sqrt{a} b \sqrt{a}$ and $a \mid b$ when $ab = ba$.

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- See how much structure we can derive from that.
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- But it can be recovered with one additional assumption.

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V's with $\omega(c) \leqslant 1 \ \forall \omega \implies c \leqslant 1$ are *order unit spaces*.

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Resistance to noise: $a \mapsto a \,\&\, b$ is continuous.

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– Not enough to characterise $b \mapsto \sqrt{a} b \sqrt{a}$.

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Proof: By Koecher-Vinberg theorem it suffices to show homogeneity + self-duality.

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- $\Rightarrow$ Positive cone of V is homogeneous.

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- $\Rightarrow$ The order unit space spanned by two atomic effects can now be shown to have a *strictly convex* cone.
- $\Rightarrow$ Ito & Lourenco (2017): A fin.dim. OUS with homogeneous and strictly convex cone is order-isomorphic to a spin factor. Hence, is self-dual.
- $\Rightarrow$ We can extend this self-duality for the subspaces to whole space.

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Sequential Measurement Characterises Quantum Theory

A locally tomographic GPT of sequential effect spaces embeds into $\mathbf{CStar}_{\mathrm{CP}}$.

Minimality of conditions

The full theorem

Let V be a finite-dimensional order unit space with $E = \{a \in V \; ; \; 0 \leqslant a \leqslant 1\}$ and $\& : E \times E \to E$ such that

- $a \,\&\, (b + c) = a \,\&\, b + a \,\&\, c$ and $a \mapsto a \,\&\, b$ is continuous,
- $1 \,\&\, a = a$ and $a \,\&\, b = 0 \implies b \,\&\, a = 0$,
- $a \,|\, b \implies a \,\&\, (b \,\&\, c) = (a \,\&\, b) \,\&\, c$,
- $a \,|\, b \implies a \,|\, b^{\perp}$ and $a \,|\, b, c \implies a \,|\, (b + c)$, $a \,|\, (b \,\&\, c)$,
- it has locally tomographic composite $V \otimes V$,
- and $(a_1 \otimes a_2) \,\&\, (b_1 \otimes b_2) = (a_1 \,\&\, b_1) \otimes (a_2 \,\&\, b_2)$.

Then V is a C*-algebra.

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- it has locally tomographic composite $V \otimes V$,
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Then V is a C*-algebra.

Minimality of conditions

The full theorem

Let V be a finite-dimensional order unit space with $E = \{a \in V \ ; \ 0 \leqslant a \leqslant 1\}$ and $\& : E \times E \to E$ such that

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Conclusion

Finite-dimensional order unit space
Continuous sequential product

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Euclidean Jordan algebra

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$\Rightarrow$ **Sequential Measurement characterises Quantum Theory**

Comparison to other reconstructions

- Don't need to refer to concept of pure states/effects.
- Don't have any pure transitivity axiom.
- Don't have any filter axiom.
- Local tomography only used to select complex algebras from the Jordan algebras.
- Axioms are not specific to finite dimensional systems.

Generalising generalised probabilistic theories

Primer on category theory

A *category* consists of *objects* $A, B, \ldots$ and *morphisms* $f : A \to B$ that compose associatively. For each A we have $\mathrm{id}_A : A \to A$.

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A *functor* $F : \mathbf{C} \to \mathbf{D}$ maps objects $A \mapsto F(A)$ and morphisms $(f : A \to B) \mapsto F(f) : F(A) \to F(B)$ such that $F(f \circ g) = F(f) \circ F(g)$, and $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$.

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A functor is *faithful* when it is injective on morphisms.

Generalized Probabilistic Theories

A GPT can be seen as a category.
They consist of

- systems $A, B, C, \ldots$,
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Special operations:

- *States* $\mathrm{St}(A) := \{\omega : I \to A\}$
- *Effects* $\mathrm{Eff}(A) := \{p : A \to I\}$
- $p \circ \omega$ is probability that p holds on state ω

From GPTs to effectuses

Problem with GPTs:

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- Can't describe deterministic models.
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Solution: allow more general sets of scalars $\{s : I \to I\}$.
Result: effectus theory.

Partial commutative monoids

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Definition

A *partial commutative monoid* (PCM) $(X, \varovee, 0)$ is a set X with a *partial* associative commutative operation $\varovee$ with unit 0.

$$(x \varovee y) \varovee z = x \varovee (y \varovee z) \qquad x \varovee y = y \varovee x \qquad x \varovee 0 = x$$

Write $x \perp y$ when $x \varovee y$ is defined.

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- Let **Cstar** be category of unital C*-algebras with positive subunital maps.
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Pfn and **Cstar** are examples of *PCM-enriched categories*:

$$(f \ovee g) \circ h = (f \circ h) \ovee (g \circ h) \qquad h \circ (f \ovee g) = (h \circ f) \ovee (h \circ g)$$

Effect algebras

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An *effect algebra* is a PCM $(E, \ovee, 0, 1)$ with special element 1 such that for each $a \in E$:

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Examples

- $[0,1]$ with $a^{\perp} := 1 - a$.
- A Boolean algebra: $a \perp b$ when $a \wedge b = 0$ and then $a \ovee b = a \vee b$. $a^{\perp}$ is the regular negation.
- $\mathbf{Cstar}(\mathbb{C}, \mathfrak{A}) \cong [0,1]_{\mathfrak{A}}$ with $a^{\perp} := 1 - a$.

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- $\mathbf{Cstar}^{\mathrm{op}}$: $I = \mathbb{C}$, $\mathbf{Cstar}^{\mathrm{op}}(\mathfrak{A}, I) \cong [0, 1]_{\mathfrak{A}}$.

Relating effectus to GPTs

Let $\mathbf{C}$ be an effectus, and $A \in \mathbf{C}$.

- Effects $\mathrm{Eff}(A) := \mathbf{C}(A, I)$ form effect algebra.
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- $\mathbf{Pfn}(I, I) \cong \{0, 1\}$ & $\mathbf{Cstar}^{\mathrm{op}}(I, I) \cong [0, 1]$.
- In general: $\mathbf{C}(I, I)$ is effect algebra.
- But also has a 'multiplication' given by composition $I \xrightarrow{s} I \xrightarrow{t} I$.

Scalars in effectus

Definition

An *effect monoid* $(M, \ovee, 0, 1, \cdot)$ is an effect algebra with associative distributive multiplication:

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Examples:

- $[0, 1]$.
- Any Boolean algebra: $a \ovee b := a \vee b$, $a \cdot b := a \wedge b$.
- $\{f : X \to [0, 1] \text{ continuous}\}$ for a compact Hausdorff space X (i.e. unit interval of commutative unital C*-algebra).

Effectus over an effect monoid

Let $\mathbf{C}$ be effectus, $M := \mathbf{C}(I, I)$ an effect monoid and $A \in \mathbf{C}$.

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Any effect monoid is the set of scalars of some effectus
Corollary: There are some weird effectuses out there

σ-PCMs

- $(\mathbb{R}, +, 0)$ does not just have *finite* sums.
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Definition (informal)

A σ-PCM is a PCM where a countable sum exists iff all the finite subsums exist.

Examples:

- **Pfn**(A, B).
- Let **Wstar** be category of von Neumann algebras with normal positive subunital maps. Then **Wstar**$(\mathfrak{A}, \mathfrak{B})$ is σ-PCM.

In fact: **Pfn** and **Wstar** are *σ-PCM enriched*.

σ-effectus

Definition

σ-effectus is σ-PCM-enriched effectus with countable coproducts.

Examples:

- **Pfn**
- $\mathbf{Wstar}^{\mathrm{op}}$

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It turns out that σ-effectuses are way more well-behaved.

σ-effect monoids

Proposition

Eff(A) in a σ-effectus is ω-complete,
i.e. increasing sequences $a_1 \leqslant a_2 \leqslant \ldots$ have suprema.

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Theorem (Westerbaan, Westerbaan & vdW, LICS'20)

An ω-directed-complete effect monoid M embeds into $M_1 \oplus M_2$
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Corollary

Scalars in a σ-effectus are commutative.

Normalisation in σ-effectuses

Theorem

Let $\mathbf{C}$ be a σ-effectus with $M = \mathbf{C}(I, I)$.
The following are equivalent.

- States in $\mathbf{C}$ can be normalized.
- Non-zero scalars are epi.
- M has a 'division' operation.
- M has no zero divisors ($a \cdot b = 0 \implies a = 0$ or $b = 0$).
- M is irreducible ($M_1 \oplus M_2 = M \implies M_1 = 0$ or $M_2 = 0$).

Furthermore, if any and thus all these conditions hold then
$M \cong \{0\}$, $M \cong \{0, 1\}$ or $M \cong [0, 1]$.

Dichotomy between deterministic and probabilistic models

Hence: σ-effectuses with normalization come in three types:

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When appropriate operational *separation properties* are satisfied, we can say even more about these latter two cases.

Separation properties

Definition

An effectus $\mathbf{C}$ has **state-separation** when $f \circ \omega = g \circ \omega$ for all states $\omega \in \mathrm{Eff}(A)$ implies $f = g$ for any $f, g : A \to B$.
(i.e. iff St: $\mathbf{C} \to \mathbf{WMod}_M$ faithful)

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Non-Example

Category of effect algebras $\mathbf{EA}^{\mathrm{op}} \cong \mathbf{EMod}^{\mathrm{op}}_{\{0,1\}}$.
Let $P(\mathscr{H}) \in \mathbf{EA}$ be the projections on a Hilbert space $\mathscr{H}$ with $\dim(\mathscr{H}) > 2$. Then Kochen-Specker theorem says $\mathrm{St}(P(\mathscr{H})) = \{0\}$.

Classical deterministic effectuses

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Hence, such effectuses are entirely classical.

Corollary

Nonclassical σ-effectuses w/ normalisation **must** have scalars $[0, 1]$.

Convex embeddings of effectuses

A similar sort of embedding holds in the probabilistic setting.

Definition

A **Banach** order unit space is an OUS that is complete in its canonical norm.

A σ-OUS is one that has countable directed suprema.

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Theorem

Let $\mathbf{C}$ be a σ-effectus with $\mathbf{C}(I, I) \cong [0, 1]$ and effect-separation.
Then there is a faithful morphism of σ-effectuses
$F : \mathbf{C} \to \sigma\mathbf{BOUS}^{\text{op}}$, the category of Banach σ-OUSes.

Hence: Probabilistic effectuses embed into the well-studied land of real ordered vector spaces.

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Summary

- A non-trivial σ-effectus with normalisation is either deterministic or probabilistic.
- When it is deterministic and has state-separation then it embeds into **Pfn** and $\mathbf{BA}^{\mathrm{op}}$, and hence is classical.
- When it is probabilistic and has effect-separation it embeds into $\mathbf{BOUS}^{\mathrm{op}}$, and hence reduces to a standard GPT-like object.
- So we managed to go from an abstract categorical framework to concrete well-studied standard settings.

Further work

- Can impose additional conditions on effectus to constrain possibilities.
- Use this to reconstruct quantum theory ‘categorically’.
- Includes infinite-dimensional systems
- See my thesis!

Thanks for listening!