Quantum Theory from First Principles
Lecture 2

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Previously

- We asked the question of how quantum theory can be derived from first principles.
- We covered many classic results: Wigner, Stone, Gleason, Jordan-von Neumann, Gelfand-Naimark, Piron, Sòler.
- We saw that the main question is deriving the structure of observables.

Today

- Ordered vector spaces
- The modern ‘operational approach’.
- Generalised probabilistic theories (GPTs).
- Composite systems and local tomography.

Ordered vector spaces

Definition

A real vector space V is *ordered* when it has a partial order $\leqslant$ satisfying

- $v \leqslant v'$ implies $v + w \leqslant v' + w$ for all $w \in V$.
- $v \leqslant v'$ implies $\lambda v \leqslant \lambda v'$ for all $\lambda \in \mathbb{R}_{\geqslant 0}$.

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Note: the order is completely determined by the *positive cone* $V^+ := \{v \in V \; ; \; v \geqslant 0\}$ via $v \leqslant w \iff w - v \geqslant 0$.

Ordered vector spaces examples

Example

Recall $A \in B(\mathscr{H})$ positive when $\langle v, Av \rangle \geqslant 0$ for all $v \in B(\mathscr{H})$.
$B(\mathscr{H})_{\mathsf{sa}}$ is an ordered vector space with $A \leqslant B$ iff $B - A$ is positive (if $\langle v, Av \rangle \leqslant \langle v, Bv \rangle$ for all $v \in \mathscr{H}$).

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Example

For a C*-algebra $\mathfrak{A}$ set $a \geqslant 0$ when $a = bb^*$ for some $b \in \mathfrak{A}$. Then $\mathfrak{A}_{\mathsf{sa}}$ is an ordered vector space.

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For a formally real Jordan algebra E set $a \geqslant 0$ when $a = b^2$ for some $b \in E$. Then E is an ordered vector space.

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In particular, for $E = S_n = \mathscr{H} \oplus \mathbb{R}$ a spin factor, we have $(v, t) \geqslant 0$ iff $t^2 \geqslant \langle v, v \rangle$.

Order isomorphism and homogeneity

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$B(\mathscr{H})_{\mathrm{sa}}$ is homogeneous (wrt operator norm topology): $A \in B(\mathscr{H})_{\mathrm{sa}}$ is in the interior positive cone iff $A \geqslant \epsilon 1$ for some $\epsilon > 0$ and thus A is invertible. For such A and B set $\Phi_{A,B}(C) = \sqrt{B}\sqrt{A^{-1}}C\sqrt{A^{-1}}\sqrt{B}$.

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(Similar constructions works for C*-algebras and formally real Jordan algebras)

Self-duality

Definition

Let V be an ordered v.s. with positive cone C and inner product $\langle \cdot, \cdot \rangle$. Its *dual cone* is $C^* := \{v \in V \ ; \ \langle v, w \rangle \geqslant 0 \text{ for all } w \geqslant 0\}$.

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Proposition

Let E be a fin.dim. real Jordan algebra. Then E is formally real iff E is Euclidean. Furthermore, E is self-dual.

From ordered vector spaces to Jordan algebras

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Recall that simple EJAs are either $M_n(\mathbb{F})$ with $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ or are spin-factors, so this brings us tantalizingly close to quantum theory.

Order units

Definition

Let V be an ordered vector space. An *order unit* is $u \in V$ such that for each $v \in V$ we have $-nu \leqslant v \leqslant nu$ for some $n \in \mathbb{N}$.
We write $[0, u]_V := \{v \in V \; ; \; 0 \leqslant v \leqslant u\}$ for the *unit interval*.

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Examples

The identity is an order unit of $B(\mathscr{H})_{\mathrm{sa}}$.
If a C*-algebra $\mathfrak{A}$ is unital, then the unit is an order unit of $\mathfrak{A}_{\mathrm{sa}}$.
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Definition

An order unit u is *Archimedean* when $nv \leqslant u$ for all $n \in \mathbb{N}$ implies $v \leqslant 0$.
An ordered vector space with an Archimedean order unit is called an *order unit space* (OUS).

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All examples above are order unit spaces.

Order units cont.

Non-example

Let $V = \mathbb{R}^2$ be equipped with lexicographic order: $(a, b) \geqslant 0$ iff $a > 0$ or $a = 0$ and $b \geqslant 0$. Then $(1, 0)$ is an order unit, but $(1, 0) \geqslant n(0, 1)$ so is not Archimedean.

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Proposition

An order unit u is Archimedean if and only if $\|\cdot\|_v$ is a proper norm and the positive cone is closed in its topology.

States on ordered vector spaces

Definition

A *state* on an ordered vector space V with order unit u is a positive linear map $\omega : V \to \mathbb{R}$ satisfying $\omega(u) = 1$.
Denote space of states by $\mathrm{St}(V)$.

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For $(B(\mathscr{H})_{\mathrm{sa}}, 1)$ with $\mathscr{H}$ finite-dimensional, each state $\omega : B(\mathscr{H}) \to \mathbb{R}$ corresponds to a density operator ρ via $\omega(a) = \mathrm{tr}(a\rho)$.

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Similar results hold for all self-dual OUSes.

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Proposition

An ordered vector space V with order unit u is an OUS if and only if the states *order-separate* the points:
$\omega(a) \leqslant \omega(b)$ for all $\omega \in \mathrm{St}(V)$ iff $a \leqslant b$.

Order unit algebras

Example

Let X be a compact Hausdorff space and define
$C(X) := \{f : X \to \mathbb{R}\ ;\ f \text{ continuous}\}$.
Then $C(X)$ is an OUS with $f \geqslant 0$ iff $f(x) \geqslant 0$ for all x;
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Kadison's representation theorem, 1951

Let V be an OUS with a bilinear operation $\cdot : V \times V \to V$ such that $v \cdot w \geqslant 0$ whenever $v, w \geqslant 0$.
Then there exists a compact Hausdorff space X and a order and algebra embedding $\Phi : V \to C(X)$ s.t. $\Phi(V)$ is dense in $C(X)$.

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Corollary

The only fin.dim. OUSes with positivity preserving bilinear operation are $\mathbb{R}^n$ with pointwise order and product.

Enough abstract vector spaces, let’s get operational!

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- In relativity: clocks, rods, events, observers, but a priori not the invariant interval.
- Entropy is a priori an abstract quantity, but via Shannon information theory can be given an operational interpretation.
- Measurement probabilities are operational: ‘prepare this state, apply this transformation, do this measurement, then record the probability of observing a certain outcome’.

GPTs/OPTs

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- A measurement of a system is represented by a collection of *effects* $a_1, a_2, \ldots, a_k \in \mathrm{Eff}(A)$.
- The probability that the outcome associated to a_j is observed when system is in state ω is denoted by $\omega(a_j) \in [0, 1]$, and we have $\sum_j \omega(a_j) = 1$.

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- This makes $\mathrm{St}(A)$ a convex set.
- We have $(p\omega_1 + (1-p)\omega_2)(a) = p\omega_1(a) + (1-p)\omega_2(a)$.
- Similarly define $pa_1 + (1-p)a_2$ for effects $a_1, a_2 \in \mathrm{Eff}(A)$. This makes $\mathrm{Eff}(A)$ a convex set.
- We have $\omega(pa_1 + (1-p)a_2) = p\omega(a_1) + (1-p)\omega(a_2)$.

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- Then ω and ω' are *operationally indistinguishable*.
- We hence set $\omega = \omega'$.
- Similarly if $\omega(a) = \omega(b)$ for all $\omega \in \mathrm{St}(A)$ then we require $a = b$.

Summary of GPT framework

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- P is 'determining': for $\omega_1 \neq \omega_2$ there is some $a \in \mathrm{Eff}(A)$ such that $P(\omega_1, a) \neq P(\omega_2, a)$ and for $a_1 \neq a_2$ there is $\omega \in \mathrm{St}(A)$ such that $P(\omega, a_1) \neq P(\omega, a_2)$.

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- Transformations $f : A \to B$ are affine maps $f : \mathrm{St}(A) \to \mathrm{St}(B)$ or equivalently, affine maps $f : \mathrm{Eff}(B) \to \mathrm{Eff}(A)$.

Example: Quantum theory as GPT

- For a system A we have a fin. dim. complex Hilbert space $\mathscr{H}_A$.
- $\mathrm{Eff}(A) = [0, 1]_{B(\mathscr{H}_A)}$.
- $\mathrm{St}(A) = \mathrm{DO}(B(\mathscr{H}_A))$.
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Generalised Gleason's theorem

For every affine $\omega : [0,1]_{B(\mathscr{H}_A)} \to [0,1]$ with $\omega(0) = 0, \omega(1) = 1$, there is a unique density operator ρ such that $\omega(a) = \mathrm{tr}(\rho a)$.

Example: Classical theory as GPT

- Systems correspond to natural numbers n.
- $\mathrm{Eff}(n) = [0,1]^n$.
- $\mathrm{St}(n) = \Delta^n := \{(p_1, \ldots, p_n) \in \mathbb{R}^n_{\geqslant 0}\ ; \sum_i p_i = 1\}$.
- $P(\omega, a) = \sum_i \omega_i a_i$.
- Transformations are affine maps $f : \Delta^n \to \Delta^m$.

Relation to ordered vector spaces

- For system A define its *associated vector space* V_A as the space of formal linear combinations $\sum_i \lambda_i a_i$ where $a_i \in \mathrm{Eff}(A)$ and $\lambda_i \in \mathbb{R}$, modulo equality among all states:

$$\sum_i \lambda_i a_i \sim \sum_j \mu_j a'_j \iff \sum_i \lambda_i \omega(a_i) = \sum_j \mu_j \omega(a'_j) \text{ for all } \omega \in \mathrm{St}(A)$$

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- $\text{Eff}(A)$ embeds as a convex subset in V_A. Allows us to define $a + b$ for $a, b \in \text{Eff}(A)$.
- $\omega \in \text{St}(A)$ give linear positive maps $\omega : V_A \to \mathbb{R}$.

Coarse graining and causality

Definition

A measurement $a_1, \ldots, a_n \in \mathrm{Eff}(A)$ is a *coarse graining* of a measurement $b_1, \ldots, b_k \in \mathrm{Eff}(A)$ if there exists a partition $S_1, \ldots, S_k$ of $\{1, \ldots, n\}$ such that $b_i = \sum_{j \in S_i} a_j$ for all i.

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NOTE: Uniqueness of deterministic effect is related to causality.

Improved GPT framework

- For each system A we have order unit space V_A.
- Effects are convex subsets $\mathrm{Eff}(A) \subseteq [0,1]_{V_A}$ containing 0 and 1.
- States are order-separating convex subset $\mathrm{St}(A) \subseteq \mathrm{St}(V_A)$ $(\mathrm{St}(V_A) := \{\omega : V_A \to \mathbb{R}\ ;\ \omega(1) = 1, \forall a \geqslant 0 : \omega(a) \geqslant 0\})$
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No-restriction hypothesis

All mathematically definable effects and states are physically realisable.

$\mathrm{Eff}(A) = [0,1]_{V_A}$
$\mathrm{St}(A) = \mathrm{St}(V_A)$.

Finite tomography

Definition
A system satisfies *finite tomography* when each state is characterised by a finite number of different measurements.

$$\exists a_1, \ldots, a_n \in \mathrm{Eff}(A) \forall \omega_1, \omega_2 \in \mathrm{St}(A) : \omega_1(a_i) = \omega_2(a_i) \iff \omega_1 = \omega_2$$

Proposition
A system A satisfies finite tomography iff V_A is finite dimensional.

We call $\dim V_A$ the *tomographic dimension* of A.

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Composite systems

For each pair of systems A and B there is a system C such that there is a bilinear map $\otimes : V_A \times V_B \to V_C$, such that $1_A \otimes 1_B = 1_C$ and $a \otimes b \geqslant 0$ when $a \geqslant 0$, and $b \geqslant 0$.

Local tomography

Definition

A GPT has *local tomography* when local measurements determine states: Let $\omega_1, \omega_2 \in \mathrm{St}(A \otimes B)$ with $\omega_1(a \otimes b) = \omega_2(a \otimes b)$ for all $a \in \mathrm{Eff}(A)$ and $b \in \mathrm{Eff}(B)$. Then $\omega_1 = \omega_2$.

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Proposition

A GPT with finite tomography has local tomography when $\dim V_{A \otimes B} = \dim V_A \; \dim V_B$.

Example

Quantum theory and classical theory both satisfy local tomography.

Non-Example

Real quantum theory, where we replace complex Hilbert spaces by real ones does *not* have local tomography.

Example: Jordan algebras as GPT

- The V_A's are Euclidean Jordan algebras.
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- The V_A's are Euclidean Jordan algebras.
- States and effects defined by no-restriction hypothesis.
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This GPT has no inherent notion of composite system.
This might be one of the main reasons the universe is described by complex Hilbert spaces instead of EJAs.

Wigner's theorem in GPTs

Definition

A transformation $\Phi : A \to B$ in a GPT is *reversible* if there is a transformation $\Psi : B \to A$ such that $\Phi \circ \Psi = \mathrm{id}_B$ and $\Psi \circ \Phi = \mathrm{id}_A$.

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Proposition

In the 'quantum GPT' of completely-positive trace-preserving maps between systems of type $M_n(\mathbb{C})_{\mathrm{sa}}$, a transformation $\Phi : M_n(\mathbb{C})_{\mathrm{sa}} \to M_n(\mathbb{C})_{\mathrm{sa}}$ is reversible iff there exists a unitary $U \in M_n(\mathbb{C})$ such that $\Phi(A) = UAU^{\dagger}$.

Some modern reconstructions of quantum theory

Next time...

- Lucien Hardy, 2001:
  Quantum Theory From Five Reasonable Axioms.
- Chiribella, D'Ariano, Perinotti, 2011:
  Informational derivation of quantum theory.
- Masanes & Müller, 2011:
  A derivation of quantum theory from physical requirements.
- Barnum, Müller, Ududec, 2014: *Higher-order interference and single-system postulates characterizing quantum theory*

Summary of the lecture

- Ordered vector spaces and relation to quantum mechanics.
- GPTs as general framework for studying alternative theories.